THE
MATHEMATICAL GAZETTE

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SYMMETRIC MATRICES AND QUADRATIC FORMS.

BY M. F. EGAN.

THE theorems that follow are mostly well known ; the object of this note is to show that they all follow simply from a single argument, whereas they are usually reached from different starting-points.

Let $F_n(x) = \Sigma\Sigma a_{pq}x_p x_q$ be a quadratic form in n variables x_k, and let $D_n = | a_{pq} |_n$ be its discriminant. If we put each of the variables $x_{p+1}, x_{p+2}, \dots, x_n$ equal to zero, we get the form $F_p(x)$ in p variables, and its discriminant D_p.

The theorems are these :

I. If D_n is of rank r, the form F_n can be expressed as the sum of r square terms, $\Sigma b_k y_k^2$, the r variables y_k being independent linear combinations of the x_p.

II. A symmetric matrix of rank r has one non-zero principal minor of order r.

III. If a symmetric matrix has a non-zero principal minor M of order r, and every principal minor of orders $r+1$ and $r+2$ which contains M vanishes, the matrix is of rank r.

IV. Suppose this minor M to be D_r. (We can always ensure this by renumbering the suffixes.) Then we can renumber the suffixes $1, 2, \dots, r$ in such a way that no two consecutive terms of the sequence

$$(S) \quad 1, D_1, D_2, \dots, D_r$$

vanish together.

V. If all the terms of (S) are different from zero, then the numbers of positive and negative coefficients respectively in the reduced form $\Sigma b_k y_k^2$ will be equal to the numbers of continuations and of changes of sign in (S). (This, of course, on the supposition that only real numbers are involved.) In particular, if all the terms of (S) are positive, the form F_n is non-negative, and is positive-definite when considered as a form in the r variables y_k. Conversely, if F_n is non-negative and of rank r, and $D_r \neq 0$, then every principal minor contained in D_r is positive.

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VI. If one of the terms of the sequence (S), say D_p, vanishes, then D_{p-1} and D_{p+1} have opposite signs, and the rule given in V holds provided that we regard the passage from D_{p-1} to D_{p+1} as involving both a change of sign and a continuation of sign.

VII. The matrix A of a real non-negative form of rank r can be expressed in the form $T'T$, where T is a real matrix of rank r. And conversely, if T is any real $m \times n$ matrix of rank r, $T'T$ is the matrix of a non-negative form of rank r.

The proof depends on two observations. First, renumbering the suffixes changes a principal minor of D_n into a principal minor, since it involves a certain number of interchanges of columns and an equal number of interchanges of corresponding rows; neither the value nor the sign of the minor is altered, and its axis still lies along the axis of D_n. Secondly, the values of all the D_p are unaltered by a "triangular" substitution of the form

$$y_k = x_k + \text{a linear form in } x_{k+1}, x_{k+2}, \ldots \quad (k = 1, 2, 3, \ldots, n).$$

In effect, D_n is unaltered, since the substitution is unimodular. Again, if we put $x_{p+1}, x_{p+2}, \ldots$ all equal to zero, the corresponding variables y_{p+1}, etc., will also vanish, and we shall have a unimodular transformation of the first p x's into the corresponding y's, leaving D_p unchanged.

That being so, suppose first that there is a square term in F_n. By renumbering the suffixes, we can call it $a_{11}x_1^2$. Then we write

$$a_{11}y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n.$$

This gives

$$F_n = a_{11}y_1^2 + G,$$

where G is a quadratic form in $x_2, x_3, \ldots, x_n$.

If there is no square term, there must be at least one non-zero product term, which we can take to be $2a_{12}x_1x_2$ ($a_{11} = a_{22} = 0$, by supposition). We put

$$a_{21}z_1 = a_{21}x_1 + a_{23}x_3 + \ldots + a_{2n}x_n,$$

$$a_{12}z_2 = a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n, \quad (a_{12} = a_{21}).$$

This gives

$$F_n = 2a_{12}z_1z_2 + H,$$

where H is a quadratic form in $x_3, x_4, \ldots, x_n$.

Dealing in like fashion with G or H, and continuing the process until all the x_k have disappeared, we get eventually a form J_r in r variables, $(r \leqslant n)$, composed of terms such as either $b_k y_k^2$ or $2g_p z_p z_{p+1}$. We may suppose the suffixes to run consecutively, as for instance:

$$J_r = b_1 y_1^2 + 2g_2 z_2 z_3 + b_4 y_4^2 + \ldots.$$

If $r < n$, we complete the transformation by putting $y_s = x_s$ for the remaining variables. We have thus a "triangular" substitution of the kind indicated above, and we can find all the D_p from the form J_r. It is evident in the first place that the matrix of F_n is of rank r, since it is transformed into the matrix of J_r by a non-singular substitution. Also each product term in J_r can be expressed as the difference of two squares, giving r squares.

If $b_k y_k^2$ occurs in J_r, the only non-zero term in the last column of D_k (as derived from J_r) will be b_k at the end of the column; hence $D_k = b_k D_{k-1}$, giving the rule for the sign of b_k in this case (see V above).

If $2g_p z_p z_{p+1}$ occurs in J_r, then J_p will contain no term in z_p (z_{p+1} having been put equal to zero), hence $D_p = 0$, its last row and column being composed of zeros. Also the only non-zero terms in the last two rows or columns of

D_{p+1} will be g_p in the penultimate place in the last row and the last column; hence $D_{p+1} = -g_p^2 D_{p-1}$. Also the product $z_p z_{p+1}$ can be expressed as the difference of two squares; hence the rule given in VI.

It is clear that $D_r \neq 0$, whether the last term in J_r is y_r^2 or $z_{r-1}z_r$; and before the renumbering of the suffixes D_r had been a principal minor of D_n; whence II.

Again, if the matrix of F_n has a non-zero principal minor of order r, we can make this minor to be D_r by renumbering. We can then proceed as we did above to reduce the form by absorbing $x_1, x_2, \ldots, x_r$. If there are any more variables to be absorbed, we shall get a non-zero principal minor of order $r+1$ or $r+2$, containing D_r. If there is no such minor, there are no more terms to be absorbed, and $F_n = J_r$, which proves III. IV follows from our discussion of the values of D_k according as the kth variable in J_r is a "y" or a "z". Again, if all the terms in the sequence (S) are positive, J_r consists entirely of square terms with positive coefficients (a product term would give a negative term in the sequence, as we have seen). J_r is therefore a positive-definite form in the y_k. Conversely, if F_n is non-negative of rank r, J_r must consist only of square terms with positive coefficients; if a product $z_p z_{p+1}$ occurred, this could become negative, and by annulling the other variables in J_r we could make J_r and therefore F_n negative; hence all the terms of (S) are positive.

To prove VII: if F_n is non-negative and A is its matrix, then we can put $u_k = b_k^{\frac{1}{2}} y_k$ ($k = 1, 2, 3, \ldots, r$), and we have

$$x'Ax = u'u.$$

If $u = Tx$ is the matrix of transformation, T being $r \times n$ and of rank r, then $u'u = x'T'Tx$, whence $A = T'T$.

(If $x'Mx = 0$ for every x, M must be skew-symmetric; hence if it is symmetric, as $A - T'T$ is, it must be identically zero.)

Conversely, if T is any $m \times n$ matrix of rank r, the matrix $M = T'T$ is obviously symmetric, since $M' = M$. Also, if we put $y = Tx$, we have

$$x'T'Tx = y'y,$$

the sum of m squares. Hence the form is non-negative. It can only vanish if each of the m terms y_k vanishes, and this is equivalent to r independent conditions, hence the rank of $T'T$ is at least equal to r; but, on the other hand, it cannot exceed r, it is therefore equal to it.

Very little alteration in enunciation and proof is needed to establish the corresponding theorems for Hermitian matrices and forms. M. F. E.

GLEANINGS FAR AND NEAR.

1456. But the mathematicians had ever been remote from common humanity, as they listened to the mechanical music of the spheres and plotted the journeys of the asymptote. We may, therefore, comfortably blame the mathematical minority for the restriction imposed upon the gay and wayward undergraduate, while praising the majority of dead Dons for the liberties granted to them.—Russell Green, *Flow Gently*, *Isis* (1943), p. 92.

1457. We shall continue to make these errors until it is recognised that in practice as well as in theory, it is essential to regard the Army, Navy and Air Force as three in one and one in three. They are a triangle of forces which must be resolved into a single force directed at a decisive point.—*National News Letter*, No. 311, June 25, 1942. [Per Mr. W. J. Hodgetts.]

A NOTE ON PAN-MAGIC SQUARES.

BY NANCY CHATER AND W. J. CHATER.

ALGEBRAIC magic squares of order 4, constructed with the numbers 1, 2, ... 16, are of three main types.

From the 4×4 frame, nine 2×2 quadrants may be cut, and the sum of the numbers in these quadrants furnishes the criteria for distinguishing between the various types of magic squares.

Type I squares have 34 as total in only five of the 2×2 quadrants, that is, in the four symmetrically disposed corner quadrants and the central one.

These contain: a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; d_1, d_2, d_3, d_4 and a_3, b_4, c_1, d_2 (Fig. 1).

a_1	a_2	b_1	b_2
a_4	a_3	b_4	b_3
d_1	d_2	c_1	c_2
d_4	d_3	c_4	c_3

FIG. 1.

Type II have totals 34 in the five quadrants of Type I and also in two additional "side" quadrants, *i.e.* those containing a_4, a_3, d_2, d_1; b_4, b_3, c_2, c_1, or those containing a_2, b_1, b_4, a_3; d_2, c_1, c_4, d_3.

Type III squares have 34 as total in all nine quadrants of the frame—the four corner ones, the central one and the four side ones. Examples of the three types of 4×4 magic squares under consideration are given below in Fig. 2.

Type I.

1	12	14	7
15	6	4	9
8	13	11	2
10	3	5	16

Type II.

5	12	14	3
15	2	8	9
4	13	11	6
10	7	1	16

Type III.

2	7	12	13
16	9	6	3
5	4	15	10
11	14	1	8

FIG. 2.

In the kind of squares we consider, the five quadrants—four corner and one central—must add to 34 if the square is algebraic magic, while a square is pan-magic if it is of Type III, *i.e.* with all nine quadrants having totals 34.

The side quadrant has $a_4 + a_3 + d_2 + d_1$, as much above or below 34 as its opposite $b_4 + b_3 + c_2 + c_1$ is below or above 34. Similarly, $a_2 + b_1 + b_4 + a_3$ is as much above or below 34 as $d_2 + c_1 + c_4 + d_3$ is below or above 34.

The rule to test whether a 4×4 magic square of the types under considera-tion is pan-magic is as follows:

Find the total of the numbers in two adjacently disposed side quadrants, say those containing a_4, a_3, d_2, d_1, and a_2, b_1, b_4, a_3. If the square is magic and both these totals are 34 it is also pan-magic. If neither total or only one equals 34 the square is not pan-magic.

In Type I, $15 + 6 + 13 + 8 = 42$; $12 + 14 + 4 + 6 = 36$; neither $= 34$.

In Type II, $15 + 2 + 13 + 4 = 34$; $12 + 14 + 8 + 2 = 36$: one set $= 34$.

In Type III, pan-magic, $16 + 9 + 4 + 5 = 7 + 12 + 6 + 9 = 34$.

As the squares are all three algebraic magic ones, the numbers in the four corner and one central quadrants total 34. Thus in square I, $1 + 12 + 6 + 15 = 14 + 7 + 9 + 4 = 11 + 2 + 16 + 5 = 8 + 13 + 3 + 10 = 6 + 4 + 11 + 13 = 34$, and similarly for the squares II and III.

Magic squares of the three types described above may be obtained from

addition tables whose array is subsequently deranged according to a rule given by Lucas (vide *Unterhaltende Probleme und Spiele in mathematischer Beleuchtung*, W. Grosse, Leipzig, 1897, pp. 45 and 46).

We cite four Lucas tables using the numbers 1, 2, ... 16 (Fig. 3).

A

	1	2	3	4
0	1	2	3	4
4	5	6	7	8
8	9	10	11	12
12	13	14	15	16

B

	1	3	5	7
0	1	3	5	7
1	2	4	6	8
8	9	11	13	15
9	10	12	14	16

C

	1	2	9	10
0	1	2	9	10
2	3	4	11	12
4	5	6	13	14
6	7	8	15	16

D

	1	2	5	6
0	1	2	5	6
2	3	4	7	8
8	9	10	13	14
10	11	12	15	16

Fig. 3.

If in any such table we denote the upper fringe terms by F_1, F_2, F_3, F_4, and the side fringe terms by f_1, f_2, f_3, f_4, and adopt the symbol $F_x f_y$ as signifying $F_x + f_y$, the addition tables above may be represented by

Stage I.

	F_1	F_2	F_3	F_4
f_1	f_1F_1	f_1F_2	f_1F_3	f_1F_4
f_2	f_2F_1	f_2F_2	f_2F_3	f_2F_4
f_3	f_3F_1	f_3F_2	f_3F_3	f_3F_4
f_4	f_4F_1	f_4F_2	f_4F_3	f_4F_4

Fig. 4.

Thus a row in any one of these tables is obtained by adding the corresponding side-fringe term to the top-fringe terms.

Variants of the tables given may be obtained by permuting the terms of the top fringe keeping the side-fringe terms steady, or we may permute the side-fringe terms keeping the top-fringe ones steady, or we may permute both side- and top-fringe terms simultaneously.

Whatever addition table is used it is subsequently treated as below.

Stage 2.

f_1F_1	f_4F_3	f_4F_2	f_1F_4
f_3F_4	f_2F_2	f_2F_3	f_3F_1
f_2F_4	f_3F_2	f_3F_3	f_2F_1
f_4F_1	f_1F_3	f_1F_2	f_4F_4

Stage 3.

f_1F_1	f_4F_3	f_2F_4	f_3F_2
f_3F_4	f_2F_2	f_4F_1	f_1F_3
f_4F_2	f_1F_4	f_3F_3	f_2F_1
f_2F_3	f_3F_1	f_1F_2	f_4F_4

Stage 1, Fig. 4, shows the quantities as they appear in the addition table. Stage 2 preserves in their places all the terms of the two principal diagonals, *i.e.* f_1F_1, f_2F_2, f_3F_3, f_4F_4, and f_4F_1, f_3F_2, f_2F_3, f_1F_4, while the other eight terms are interchanged, any term of them for the one disposed symmetrically to it relative to the square's centre. Thus the interchanges are f_1F_2 with f_4F_3, f_1F_3 with f_4F_2, f_2F_1 with f_3F_4, and f_3F_1 with f_2F_4.

Stage 3 interchanges the top right-hand 2×2 quadrant of Stage 2 with the left-hand bottom one, the positions of all other terms in Stage 2 remaining unchanged. This process produces a magic square from the addition table, for it brings into each row, column and full diagonal the eight quantities $f_1, f_2, f_3, f_4, F_1, F_2, F_3, F_4$, whose sum is termed the square constant. For the squares we consider, the square constant equals the sum of all the integral numbers up to 16 divided by 4, the number of rows or columns. Thus the magic square constant, for these 4×4 squares considered, is $\dfrac{16 \times 17}{2 \times 4} = 34$.

It may be pointed out that although the fringe terms of the addition tables C and D are different, the squares they produce are intimately related. From C we obtain :

Stage 2.

1	15	8	10
14	4	11	5
12	6	13	3
7	9	2	16

Stage 3.

1	15	12	6
14	4	7	9
8	10	13	3
11	5	2	16

while from D we obtain :

Stage 2.

1	15	12	6
14	4	7	9
8	10	13	3
11	5	2	16

Stage 3.

1	15	8	10
14	4	11	5
12	6	13	3
7	9	2	16

We see that Stage 2 of both C and D are already magic squares and that Stage 3 D is the same as Stage 2 C, while Stage 3 C is the same as Stage 2 D.

We notice, too, that the Lucas process brings into the four corner 2×2 quadrants and the central one all f's and F's, so that the totals in these quadrants is the square constant 34.

By means of the fringe permutations we may obtain three essentially different patterns of Type I, two of Type II and one of Type III.

The "pattern" of square is characterised by the position in the frame of pairs of numbers, which have been termed "complementary numbers" and whose sum is half the square constant. This value is $34/2 = 17$ in our case.

Such a pair of values is marked a, a; b, b; ..., and the characteristic patterns are as given below.

Type I patterns (positions of complements):

a	*b*	*c*	*d*
e	*f*	*g*	*h*
h	*g*	*f*	*e*
d	*c*	*b*	*a*

(*a*) Symmetric about the square's centre.

a	*b*	*c*	*d*
c	*f*	*g*	*h*
a	*b*	*c*	*d*
e	*f*	*g*	*h*

(*b*) Alternate in columns.

a	*b*	*a*	*b*
c	*d*	*c*	*d*
e	*f*	*e*	*f*
g	*h*	*g*	*h*

(*c*) Alternate in rows.

a	*b*	*c*	*d*
b	*a*	*d*	*c*
e	*f*	*g*	*h*
f	*e*	*h*	*g*

(*d*) Diagonally in corner quadrants.

Examples:

1	15	8	10
12	6	13	3
14	4	11	5
7	9	2	16

(*a*)

2	13	8	11
12	7	14	1
15	4	9	6
5	10	3	16

(*b*)

2	14	15	3
7	11	10	6
12	8	5	9
13	1	4	16

(*c*)

4	10	15	5
7	13	12	2
14	8	1	11
9	3	6	16

(*d*)

As rows become columns and columns rows by merely rotating the frame, there are only three substantially different patterns of Type I, viz. (*a*), (*b*) and (*d*).

Type II has two essentially different patterns:

a	*a*	*b*	*b*
c	*c*	*d*	*d*
e	*e*	*f*	*f*
g	*g*	*h*	*h*

(*a*) Adjacent in rows.

a	*b*	*b*	*a*
c	*d*	*d*	*c*
e	*f*	*f*	*e*
g	*h*	*h*	*g*

(*b*) Means—extremes in rows.

Examples :

2	15	8	9
12	5	14	3
13	4	11	6
7	10	1	16

(*a*)

2	5	12	15
16	11	6	1
7	4	13	10
9	14	3	8

(*b*)

Adjacent in columns and means-extremes in columns are not substantially different squares from these.

Type III, which is the pan-magic type, has complements alternate in all diagonals, in the full or principal ones and in the broken ones also.

a	h	e	c
g	b	d	f
e	c	a	h
d	f	g	b

Pattern.

2	7	12	13
16	9	6	3
5	4	15	10
11	14	1	8

Example.

In the figure there are two " full " diagonals, a, b, a, b, and c, d, c, d ; and six broken ones of type a, (f, a, f) ; (g, h), (g, h), etc. If a different cut were taken from the magic parquet, some of the broken diagonals would become full ones.

Thus there are three substantially different patterns of 4×4 algebraic magic squares of Type I, two of Type II, and one of Type III, *i.e.* in all six substantially different patterns of 4×4 squares.

In the case of a pan-magic square, which we may regard as being cut from a magic parquet in which the square's numbers are regularly repeated out in all directions so as to cover the whole surface of which the 4×4 frame is but a part, we may use a row-column operator investigation to obtain suggestions for numerical connections between the values in the frame.

The investigation suggests the rule already proposed to test 4×4 squares for pan-magic quality and extends it to squares of any side. Such squares to which the extended test is applicable may be those it is possible to construct with consecutive integers, or others constructed in any way with integers omitted or with repetitions. It is a generally applicable rule.

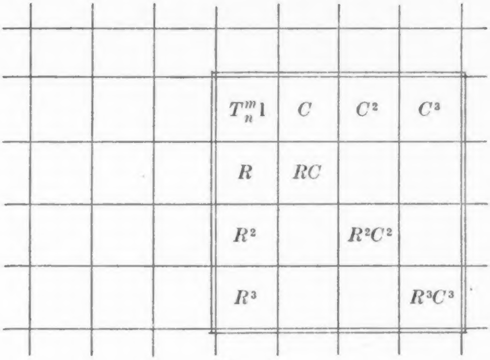


FIG. 4.

Suppose we select from the magic parquet Fig. 4 some term T_n^m of it. Make this the top left-hand term of our 4×4 frame. Then the terms of the pan-magic square are denoted as :

$1 \cdot T_n^m$, $R \cdot T_n^m$, $R^2 \cdot T_n^m$, $R^3 \cdot T_n^m$ down the column ;

$1 \cdot T_n^m$, CT_n^m, $C^2 \cdot T_n^m$, $C^3 \cdot T_n^m$ along the row ;

and by $1 \cdot T_n^m$, $RC \cdot T_n^m$, $R^2C^2 \cdot T_n^m$, $R^3C^3 \cdot T_n^m$ along the principal diagonal.

Here 1 represents an idem-operator, R a row-operator, and C a column operator. R increases the row-superscript by unity, while C increases the column-subscript by unity.

Whence we have : $$1 \cdot T_n^m + C \cdot T_n^m + C^2 \cdot T_n^m + C^3 \cdot T_n^m = S, \quad \text{..................I}$$
$$1 \cdot T_n^m + R \cdot T_n^m + R^2 \cdot T_n^m + R^3 \cdot T_n^m = S, \quad \text{...............II}$$
$$1 \cdot T_n^m + RC \cdot T_n^m + R^2C^2 \cdot T_n^m + R^3C^3 \cdot T_n^m = S, \quad \text{......III}$$

where $S =$ the magic square constant, in all our cases $= 34$.

Equating I and III and removing T_n^m, we obtain an operator equation :

$$1 + C + C^2 + C^3 = 1 + RC + R^2C^2 + R^3C^3,$$

or

$$C(R-1) + C^2(R^2-1) + C^3(R^3-1) = 0,$$

or

$$1 + C(R+1) + C^2(R^2+R+1) = 0. \quad \text{......................IV}$$

Considerations of symmetry yield also :

$$1 + R(C+1) + R^2(C^2+C+1) = 0. \quad \text{.......................V}$$

Equating IV and V, we have after simplification :

$$1 + R + C + RC = 0.$$

Now $1 + R + C + RC$ operating on T_n^m produces the block of terms in a 2×2 quadrant, $T_n^m + T_n^{m+1} + T_{n+1}^m + T_{n+1}^{m+1}$, and suggests that these four terms, or any four similarly situated ones throughout the parquet, are connected by some similar relation, no matter where $T_n{}^m$ is chosen in the parquet. Now in any 4×4 algebraic magic square, including the pan-magic ones, we

have the sum of the terms in 2×2 quadrant positions always equal to 34, provided the quadrants are corner ones or central. Whence the operator investigation suggests that for the parquet, *i.e.* the pan-magic case, every 2×2 quadrant in the 4×4 frame cut from it has values whose sum is 34. As we have seen, there are nine such 2×2 quadrants, and in a pan-magic square all nine quadrants do contain numbers whose sum is 34, in agreement with the relation suggested by the operators.

Let us examine for a moment the famous square which appeared in Albrecht Dürer's picture "Melancholie". We see at once that this square (Fig. 5) is not pan-magic, for taking the side quadrant 8, 10, 6, 12, its total is 36. Similarly, the side quadrant 6, 7, 14, 15, has total 42. The total 36 is as much above 34 as that in the oppositely disposed side quadrant 11, 5, 9, 7 $(=32)$ is below 34. Similarly, $6+7+14+15=42$ is as much above 34 as $3+2+11+10=26$ is below it.

13	3	2	16
8	10	11	5
12	6	7	9
1	15	14	4

FIG. 5.

The Dürer square is of Type I, pattern (*a*), complements being arranged symmetrically with regard to the square's centre.

An extension of the row-column operators to parquets based on pan-magic squares of different orders will be found to lead to the following relations concerning the sums of the quantities in sub-squares, whose sides are two less than the square under consideration. The rules are general for all possible pan-magic squares.

1	1
1	1

Order 4×4. Sub-square 2×2.

1	1	1
1	2	1
1	1	1

Order 5×5. Sub-square 3×3.

FIG. 6.

1	1	1	1
1	2	2	1
1	2	2	1
1	1	1	1

Order 6×6. Sub-square 4×4.

1	1	1	1	1
1	2	2	2	1
1	2	3	2	1
1	2	2	2	1
1	1	1	1	1

etc.

Order 7×7. Sub-square 5×5.

The last pattern means, for example, that if we cut from a magic parquet constructed on the basis of a 7×7 repetition a 7×7 frame, and then take any 5×5 sub-frame of it, the sum of all the values on the outside boundary of the sub-frame + twice the sum of those in the next ring + three times those in the following ring, and so on, is always constant.

If we take the square of page 206 in *Mathematical Recreations and Essays* by Rouse Ball, revised by Coxeter, viz. :

7	20	3	11	24
13	21	9	17	5
19	2	15	23	6
25	8	16	4	12
1	14	22	10	18

we have on testing it for pan-magic quality according to the above (Order 5×5),

$$7 + 20 + 3 + 9 + 15 + 2 + 19 + 13 = 88$$

and

$$2 \times 21 = 42$$

$$\underline{\underline{130}}$$

And selecting any other 3×3 square we have, say,

$$2 + 15 + 23 + 4 + 10 + 22 + 14 + 8 = 98$$

and

$$2 \times 16 = 32$$

$$\underline{\underline{130}} \text{ as required.}$$

On page 191 of his work *Mathematical Recreations*, 1944, Kraitchik gives three examples of pan-magic squares. On page 189 he states that " a pan-magic algebraic magic square is characterised by the fact that the complementary numbers lie on the diagonals in pairs whose elements separate each other ".

His three examples are :

1	8	10	15
12	13	3	6
7	2	16	9
14	11	5	4

(*a*)

1	8	11	14
13	12	2	7
6	3	16	9
15	10	4	5

(*b*)

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

(*c*)

These three squares satisfy the above " characterisation ". Thus in the middle square the full diagonals are 1, 12, 16, 5 ; 15, 3, 2, 14, alternate numbers in which add to 17. The broken diagonals are 1, 10, 16, 7 ; 13, 8, 4, 9 ; 6, 12, 11, 5 ; 15, 8, 2, 9 ; 6, 10, 11, 7 ; 13, 3, 4, 14, in all of which alternate numbers add to 17. Thus the quality characterising this square as a pan-magic one seems complete. Nevertheless the square, far from being pan-magic, is not even magic. All the rows add up each to 34, so do the two principal diagonals, and so also do the six broken diagonals, but the columns

do not add up to 34 but to 35, 33, 33, 35, respectively, beginning from the left. That the square is not pan-magic may be detected at once by means of the above rule.

The set in the side quadrant, $13+12+3+6=34$, while the adjacently disposed set $3+16+4+10=33$. To be pan-magic each set should total 34.

In the case of Kraitchik's other examples, (*a*) and (*c*) above, we have for (*a*) :

$$(12+13+2+7)=(2+16+5+11)=34,$$

and for (*c*) :

$$(14+11+5+4)=(5+16+3+10)=34,$$

both satisfying his characterisation and our rule.

If we reverse the Lucas process, using the imperfect example (*b*) above, we reach the addition table :

	0	4	6	2
1	1	5	7	3
9	9	13	15	11
10	10	14	16	12
2	2	6	8	4

which corresponds to C in Fig. 3.

This yields :

Stage 2. and Stage 3.

1	8	6	3
12	13	15	10
11	14	16	9
2	7	5	4

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

This Stage 3 square is a pan-magic one. Its side quadrants $12+13+3+6$ and $3+16+5+10$ both add to 34. Stage 3 is precisely the same as the imperfect example (*b*) above, except that the terms 12 and 13 and 5 and 4 have been interchanged. The interchanges do not affect the " characterisation ", since as much is added by interchanging 13 for 12 as is lost by interchanging 4 for 5. But these interchanges do affect the totals in the two test quadrants—at least in one of them, which is $3+16+4+10=33$ in lieu of $3+16+5+10=34$.

Reverting to the considerations embodied in the diagrams of Fig. 6, we would add the following observations :

The square constants for possible normal squares of order $n \times n$ are :

$n \times n$	4×4	5×5	7×7	etc.
S	34	65	175	etc.

$S=\frac{1}{2}n(n^2+1)$, for values for which n is possible.

To test whether a given $n \times n$ magic square is pan-magic, select from it any $(n-2) \times (n-2)$ frame whatsoever. The quantities in the frame can be

divided into rings. Those in the outer ring are added as they stand, those in the next ring are added and the total multiplied by 2; those in the following ring have their sum multiplied by 3, and so on. All these products are totalled.

This total equals $K \times S$ where K, like S, is a quantity varying with n.

We find:

	$n \times n$	4×4	5×5	6×6	7×7	8×8
	K	1	2	$3\frac{1}{3}$	5	7
or	$3K$	3	6	10	15	21

For a 7×7 pan-magic square, which is possible making use of all the whole numbers, 1, 2 ... 49, any 5×5 block of numbers selected from it multiplied by the multipliers appropriate to the special rings and added together give $K \times S$ with $K=5$, $S=175$ *i.e.* gives 875.

The numbers $3K$ (3, 6, 10 ...) are one of the familiar sets of quantities known as figurate numbers—a set whose second difference is unity. The numbers in the set are ${}^3C_1$, ${}^4C_2$, ${}^5C_3$, etc. Thus for a possible normal pan-magic square of order $n \times n$ from which we select a block $(n-2) \times (n-2)$, we have for the sums of the products from the various rings:

$$\text{Sum of the products} = \frac{1}{3} \cdot {}^{n-1}C_2 \cdot \frac{n(n^2+1)}{2}$$

$$= \frac{1}{3} \frac{(n-1)(n-2)}{1 \cdot 2} \times \frac{n(n^2+1)}{2} = \frac{1}{12} n(n-1)(n-2)(n^2+1).$$

Testing this result for the 7×7 square in Rouse Ball's work, page 206, Fig. xvii:

P Q

26	21	9	4	48	36	31
44	39	34	22	17	12	7
20	8	3	47	42	30	25
38	33	28	16	11	6	43
14	2	46	41	29	24	19
32	27	15	10	5	49	37
1	45	40	35	23	18	13

O R

Selecting the 5×5 block $OPQR$, we have:

$$9+4+48+36+31+7+25+43+19+24+29+41+46+28+3+34=427.$$

$$2 \times (22+17+12+30+6+11+16+47) = 2 \times 161 = 322. \quad 3 \times 42 = 126.$$

Total $427+322+126=875$. If the square is pan-magic this total should be:

$$\frac{1}{12} \cdot 7(7-1)(7-2)50 = \frac{7 \cdot 6 \cdot 5 \cdot 50}{12}$$

$$= 7 \times 5 \times 25 = 7 \times 125 = 875 \text{ as required.}$$

For the 6×6 square on page 211, Fig. xxiii:

28	1	26	36	8	21
3	35	7	27	23	25
34	24	22	2	29	9
4	32	19	12	39	14
13	17	15	37	5	33
38	11	31	6	16	18

In this square the numbers 10, 20, 30 are omitted, and 37, 38, 39 are used. S, the square constant, is

$$\frac{\frac{39 \times 40}{2} - (10+20+30)}{6} = \frac{780 - 60}{6} = 130 - 10 = 120.$$

We select a 4×4 sub-square.

$$\begin{aligned} 28+1+26+36+27+2+12+19+32+4+34+3 &= 224 \\ 2 \times (35+7+22+24) = 2 \times 88 &= \underline{176} \\ &\ \underline{\underline{400}} \end{aligned}$$

According to the rule the sum of the products should equal

$$\frac{1}{3}\,{}^{n-1}C_2 \,.\, S = \frac{1}{3} \,.\, {}^5C_2 \,.\, 120 = \frac{1}{3} \,.\, \frac{5 \times 4}{1 \times 2} \times 120 = 400 \text{ as required.}$$

Taking another 4×4 sub-square:

$$\begin{aligned} 22+2+29+9+14+33+18+16+6+31+15+19 &= 214 \\ \text{and} \qquad 2 \times (12+39+5+37) = 2 \times 93 &= \underline{186} \\ &\ \underline{\underline{400}} \text{ as required.} \end{aligned}$$

Thus, judged from the two tests, the square is pan-magic.

Applied to a square, each cell of which contains the value unity—pan-magic square with complete repetition—the above test leads to the following identities:

(*a*) With $2n+1$ cells in the side of the square:

$$n \,.\, 1 + (n-1) \,.\, 8 + (n-2) \,.\, 2 \,.\, 8 + \ldots + (n-r) \,.\, r \,.\, 8 + \ldots$$
$$+ [n-(n-1)](n-1) \,.\, 8 \equiv \frac{1}{3}\binom{2n}{2} \,.\, (2n+1).$$

(*b*) With $2n+2$ cells in the side:

$$. \ n \,.\, 4 + (n-1)(4+1 \,.\, 8) + (n-2)(4+2 \,.\, 8) + \ldots + (n-r)(4+r \,.\, 8) + \ldots$$
$$+ [n+(n-1)]\,[4+(n-1)8] \equiv \frac{1}{3}\binom{2n+1}{2} \,.\, (2n+2).$$

The 4×4 square

2	8	13	21
18	16	7	3
9	1	20	14
15	19	4	6

has a square constant 44 ; 5, 10, 11, 12, 17 are omitted, other whole numbers up to 21 are included. The square constant is 44 and, as the pan-magic test prescribes, all nine quadrants total 44. The complementary numbers are alternate in diagonals, and in this case add to 22.

SUMMARY : A classification of 4×4 algebraic magic squares into three types has been made, based on the totals of certain 2×2 quadrants (adjacent side quadrants). There are three types, those in which five quadrants each total to S (the square constant), those in which seven each total to S, and those in which all nine quadrants each total to S. There are three essentially different patterns of Type I, two of Type II and one of Type III, the 4×4 pan-magic type.

With the aid of row-column operators, a general rule has been stated which may be used to check the pan-magic quality of all magic squares of any order.

If the square being tested is of order $n \times n$, select from it haphazard any sub-square $(n-2) \times (n-2)$. Find the total of the quantities in the outer ring of the sub-square, twice the total of those in the next ring, three times those in the following ring, and so on. The grand total for the sub-square, got by adding the quantities contributed from the rings, equals $\frac{1}{3}\binom{n-1}{2}S$ where $\binom{n-1}{2}$ is a combination symbol and S equals the magic square constant. This rule has been checked on examples given in works by Kraitchik and Rouse Ball.

N. C.

W. J. C.

1458. When I first went up to Cambridge (1877), I confounded the Circle at Infinity with the Circular Points at Infinity till someone drew a circle for me and put two circular points in it like two eyes in a very fat face, and then added the Line at Infinity just where the mouth would come. And now I cannot go to Infinity without seeing this round face grinning at me as the Cheshire Cat grinned at Alice when she was in Wonderland.

In those days there were old Dons at Cambridge who rampaged like mad bulls, if you just waved red rags at them. If the Don was Mathematical, you waved the Method of Projections; if he was Classical, you waved Archaeology. With the Method of Projections a short proof was substituted for a long proof; but the old men had always used the long proof, and were indignant that the same results should be obtained so easily; and they had influence enough to get the easy proof prohibited in the Mathematical Tripos.—Cecil Torr, *Small Talk at Wreyland* (1926), p. 94.

Torr was a Senior Optime in 1880 (and also took the Classical Tripos): the quotation is an interesting light on undergraduate opinion of the time. [Per Mr. P. Fraser.]

STABILITY OF EQUILIBRIUM.

By F. Underwood.

1. This article contains a discussion of the problem considered by Routh* and other writers under the general heading "Rocking Stones", but is restricted to cases which can be treated as problems in two dimensions. Thus Routh writes: "A perfectly rough heavy body rests in equilibrium on a fixed surface; it is required to determine whether the equilibrium is stable or unstable. We shall suppose the body to be displaced in a plane of symmetry so that the problem may be considered to be one in two dimensions."

The following cases are considered:

(i) Heavy body resting on a horizontal plane.

(ii) Heavy body resting on a fixed surface with the common normal vertical in the position of equilibrium.

(iii) Heavy body resting on a fixed surface with the common normal inclined to the vertical at an angle α in the position of equilibrium.

These cases are all considered by various methods either by Routh or Minchin,† and (in a sense) Routh's general method ‡ may be said to include all three cases. This article uses one general method to give a complete treatment of (i) and (iii), and indicates how all results required in (ii) can be obtained as special cases of (iii). The following points may be noted:

(*a*) The present work is extended beyond the results given by Routh in the general case (iii), for it gives the results necessary for the third "critical" case § as well as those for the first and second critical cases.

(*b*) The special case (ii) deduced from (iii) serves to correct errors, or, at least, very misleading statements by Minchin.

(*c*) A simple general theorem is given which covers all the critical cases of (i); this is not stated explicitly by Routh or Minchin.

(*d*) Examples are given showing how critical cases under (i) can be treated without any preliminary bookwork, or even more simply by using the general theorem in (*c*) above.

All cases of rolling displacement depend upon the fact that the length of arc s is the same for both bodies when measured from the point of contact in a position of equilibrium to that in an adjacent position, and in (ii) and (iii) the present method requires s as the independent variable, but in (i), where only a single angle ψ (in the usual sense) appears, the results are simplified by the use of ψ as the independent variable, though if required with s as the independent variable, they may be written down as special cases of (iii).

2. In (i) let A be the point of contact with the horizontal plane of the curve of the cross-section made by the vertical plane through G, the centre of gravity of the heavy body, and let axes Ax, Ay be respectively the tangent and the upward normal to this curve at A. Let (x, y) be the coordinates of

* E. J. Routh, *Treatise on Analytical Statics*, 2nd ed. (1896 and 1909), Vol. I, pp. 175–180.

† G. M. Minchin, *Treatise on Statics*, 4th ed. (1889), Vol. II, pp. 134–139; 5th ed. (1915), pp. 86–91.

‡ *Ibid.*, pp. 178–180.

§ "Critical case" is the term used by Minchin, while Larmor writes "On critical or 'apparently neutral' equilibrium".

B, a point on this curve close to A, and let (X, Y) be the coordinates of any point on the tangent at B. The equation of this tangent is

$$X \sin \psi - Y \cos \psi = x \sin \psi - y \cos \psi,$$

where $\tan \psi$ is the value of dy/dx at B.

If $AG = h$, the coordinates of G are $(0, h)$, where h is fixed.

Hence $\quad z =$ length of the perpendicular from G upon the tangent at B

$$= x \sin \psi + (h - y) \cos \psi.$$

The potential energy of the body in a displaced position, *i.e.* with B in contact with the horizontal plane, is Wz, where W is the weight of the body. If $x', x'', \ldots, x^{(r)}$ denote $dx/d\psi, d^2x/d\psi^2, \ldots, d^rx/d\psi^r$,

$$x' = \frac{dx}{ds} \cdot \frac{ds}{d\psi} = \rho \cos \psi, \quad y' = \frac{dy}{ds} \cdot \frac{ds}{d\psi} = \rho \sin \psi.$$

$$\begin{aligned} \text{Hence} \quad z' &= x \cos \psi - (h - y) \sin \psi + \rho (\cos \psi \sin \psi - \sin \psi \cos \psi) \\ &= x \cos \psi - (h - y) \sin \psi. \\ z'' &= -x \sin \psi - (h - y) \cos \psi + \rho (\cos^2 \psi + \sin^2 \psi) \\ &= \rho - x \sin \psi - (h - y) \cos \psi \\ &= \rho - z. \end{aligned}$$

Thus $$z^{(n)} = \rho^{(n-2)} - z^{(n-2)}. \quad \text{.................................(1)}$$

Let the suffix $_0$ refer to the position of equilibrium in which $x = 0$, $y = 0$, $\psi = 0$.

Then z is a minimum and the equilibrium is stable if $z_0^{(r)}$ vanishes for $r = 1, 2, \ldots, (2n - 1)$, and $z_0^{(2n)} > 0$. Using equation (1) and the initial values $z_0 = h$, $z_0' = 0$, $z_0'' = \rho_0 - h$, these are equivalent to the conditions

$$0 = \rho_0 - h = \rho_0' = \rho_0'' = \ldots = \rho_0^{(2n-3)}, \quad \text{........................(2)}$$

and $$\rho_0^{(2n-2)} > 0. \quad \text{....................................(3)}$$

The equilibrium is unstable if (2) holds, but (3) is replaced by $\rho_0^{(2n-2)} < 0$. Taking account of the simplest case $z_0' = 0$, $z_0'' > 0$, we can sum up by saying that for stability it is necessary and sufficient that either $\rho_0 > h$, or else that $\rho_0 = h$ and that the first of $\rho_0', \rho_0'', \ldots, \rho_0^{(r)}$ that does not vanish should have r even, say $r = 2n - 2$, and that $\rho_0^{(2n-2)}$ should be positive. [It is assumed, of course, that all these derivatives of ρ exist.] A simpler statement is that either $\rho_0 > h$, or else $\rho_0 = h$ and is a minimum value of ρ.

It is clear that in (i) the "critical" case of Minchin and Larmor is that for which $\rho_0 = h$. This may actually be regarded as a *first* critical case. A second critical case is given by $0 = \rho_0 - h = \rho_0' = \rho_0''$, and n successive critical cases are given by (2) together with (3) replaced by $\rho_0^{(2n-2)} = 0$.

3. A few simple examples of (i) are treated below.

(*a*) Perhaps the simplest non-spherical body which leads to an apparently neutral or critical case in (i) is that of a paraboloid of revolution with its axis vertical,* or a parabolic cylinder with the plane through the axes of all its principal sections vertical. Taking the principal section as $x^2 = 4ay$ and B as $(2a \tan \psi, a \tan^2 \psi)$,

$$z = h \cos \psi + a \sin \psi \tan \psi.$$

$$z' = -h \sin \psi + a \sin \psi (1 + \sec^2 \psi).$$

$$z'' = -h \cos \psi + a (\cos \psi + \sec \psi + 2 \sec \psi \tan^2 \psi).$$

* This example is given by Minchin, 4th ed., p. 139; or 5th ed., p. 91, Ex. 4.

Hence $z_0'=0$, $z_0''=-h+2a$, and the first critical case arises when $h=2a$. In this case $z''=a\sin^2\psi\,(\sec\psi+2\sec^3\psi)$, so that $z_0'''=0$, $z_0^{\mathrm{IV}}=6a$, and the equilibrium is actually stable.

(*b*) If the principal section of the body resting on the horizontal plane is a catenary with its vertex as the point of contact with the plane, $s=c\tan\psi$, $\rho=c\sec^2\psi$ and $\rho_0=c=h$ for the first critical case. As ρ is then a minimum for $\psi=0$, the equilibrium is stable.

(*c*) If the body is a cylinder of which the curve of cross-section of the lower part in contact with the horizontal plane is part of a cardioid, with A as that end of the axis (or initial line) which is opposite to the cusp O, then, taking B as (r, θ), where $r=a(1+\cos\theta)$, $\psi=3\theta/2$ (measured from Ax), the equation of the tangent at B referred to Ax, Ay as axes is

$$X\sin\psi-Y\cos\psi=\tfrac{3}{2}a(\cos\tfrac{1}{3}\psi-\cos\psi).$$

Hence

$$z=h\cos\psi+\tfrac{3}{2}a(\cos\tfrac{1}{3}\psi-\cos\psi),$$

$$z'=-h\sin\psi-\tfrac{1}{2}a(\sin\tfrac{1}{3}\psi-3\sin\psi),$$

$$z''=-h\cos\psi-\tfrac{1}{6}a(\cos\tfrac{1}{3}\psi-9\cos\psi).$$

Here $z_0'=0$, $z_0''=-h+\tfrac{4}{3}a$, so that the value of h for the critical case is $4a/3$. Then

$$z''=\tfrac{1}{6}\,a(\cos\psi-\cos\tfrac{1}{3}\psi),$$

$$z'''=\tfrac{1}{18}a(\sin\tfrac{1}{3}\psi-3\sin\psi),$$

$$z^{\mathrm{IV}}=\tfrac{1}{54}a(\cos\tfrac{1}{3}\psi-9\cos\psi).$$

Hence $z_0'''=0$, $z_0^{\mathrm{IV}}=-4a/27$, and the equilibrium is unstable.

(*d*) Some interesting examples of case (i) are given when the principal curves of cross-section have certain fairly simple intrinsic equations. Thus if s is measured from A, the curves $s=h\psi+a\psi^5$ and $s=h\psi+a\sin^5\psi$ satisfy all the conditions $z_0^{(r)}=0$, $(r=1, 2, \ldots, 5)$, but $z_0^{(6)}=120a$, so each of these curves gives the first and second critical cases, but $z_0^{(5)}=0$, and the equilibrium is stable or unstable according as a is positive or negative.*

More generally, the curves $s=h\psi+a\psi^{2n}$, $s=h\psi+a\sin^{2n}\psi$, $s=h\psi+a\psi^{2n+1}$, $s=h\psi+a\sin^{2n+1}\psi$, where n is a positive integer, satisfy the conditions for [illegible] successive critical cases, but whereas the first two curves are sometimes said to give stability for displacements on one side only of the vertical,† the third and fourth curves satisfy the condition $z_0^{(2n+1)}=0$, and so give stability or instability according as a is positive or negative.

4. In the general case (iii) it is assumed that the curves of cross-section are convex to one another and that, in the position of equilibrium to be considered, the common normal at A (a point of the upper body) and C (in the lower body) is inclined to the upward vertical at an acute angle α. In the displaced position, with a common normal at B inclined to the vertical at an angle $\alpha+\phi$, the normals at A and B meet in D and the normals at C and B meet in E. For the upper body the axes used are Ax, Ay, the tangent and the upward normal at A respectively, while for the lower body the axes are CX, CY, the tangent and the downward normal at C respectively, so that for the lower body X, Y, ϕ, r correspond to x, y, ψ, ρ respectively for the upper body. It is also convenient to use $k=1/\rho$, $m=1/r$, $u=k+m$.

* If the order of the critical cases is regarded as unimportant, these results may be obtained readily by noting that the values of ρ are respectively $h+5a\psi^4$ and $h+5a\sin^4\psi\cos\psi$, so that ρ_0 is a minimum or maximum according as a is positive or negative.

† This "one-sided stability" seems to be actually instability. See § 5.

Then

$$\begin{aligned}
z &= V/W = \text{Height of } G \text{ above } C \\
&= ED\cos(\alpha+\phi) - AD\cos(\alpha+\phi+\psi) - EC\cos\alpha + h\cos(\phi+\psi) \\
&= \left(\frac{x}{\sin\psi}+\frac{X}{\sin\phi}\right)\cos(\alpha+\phi) - (y + x\cot\psi)\cos(\alpha+\phi+\psi) \\
&\qquad\qquad - (Y + X\cot\phi)\cos\alpha + h\cos(\phi+\psi) \\
&= x\sin(\alpha+\phi+\psi) - y\cos(\alpha+\phi+\psi) + h\cos(\phi+\psi) - X\sin\alpha - Y\cos\alpha \\
&= Q - X\sin\alpha - Y\cos\alpha, \quad \text{say.}
\end{aligned}$$

Let $x\cos(\alpha+\phi+\psi) + y\sin(\alpha+\phi+\psi) - h\sin(\phi+\psi) = P$.

Let $z^{(r)}$ denote $\dfrac{d^r z}{ds^r}$, so that $P' = -uQ + \cos(\alpha+\phi)$; $Q' = uP + \sin(\alpha+\phi)$.

Then $z' = uP$; $z'' = u\cos(\alpha+\phi) + u'P - u^2 Q$;

$$\begin{aligned}
z''' &= 2u'\cos(\alpha+\phi) - u(u+m)\sin(\alpha+\phi) + (u'' - u^3)P - 3uu'Q. \\
z^{\mathrm{IV}} &= \{3u'' - u(u^2+um+m^2)\}\cos(\alpha+\phi) - (5uu' + 3mu' + um')\sin(\alpha+\phi) \\
&\qquad + (u''' - 6u^2u')P - (4uu'' + 3u'^2 - u^4)Q. \\
z^{\mathrm{V}} &= \{4u''' - (9u^2 + 7um + 4m^2)u' - (u^2 + 3um)m'\}\cos(\alpha+\phi) \\
&\qquad - \{(9u+6m)u'' + um'' + 8u'^2 \\
&\qquad\qquad + 4u'm' - u(u^3 + u^2m + um^2 + m^3)\}\sin(\alpha+\phi) \\
&\qquad + (u^{\mathrm{IV}} - 10u^2u'' - 15uu'^2 + u^5)P - (5uu''' + 10u'u'' - 10u^3u')Q. \\
z^{\mathrm{VI}} &= \{5u^{\mathrm{IV}} - (19u^2 + 16um + 10m^2)u'' - (u^2 + 4um)m'' - (33u + 15m)u'^2 \\
&\qquad - (9u + 15m)u'm' - 3um'^2 \\
&\qquad\qquad + u(u^4 + u^3m + u^2m^2 + um^3 + m^4)\}\cos(\alpha+\phi) \\
&\qquad - \{(14u + 10m)u''' + um''' + (35u' + 10m')u'' + 5u'm'' \\
&\qquad\qquad - (14u^3 + 12u^2m + 9um^2 + 5m^3)u' \\
&\qquad\qquad - (u^3 + 3u^2m + 6um^2)m'\}\sin(\alpha+\phi) \\
&\qquad + (u^{\mathrm{V}} - 15u^2u''' - 60uu'u'' - 15u'^3 + 15u^4u')P \\
&\qquad - (6uu^{\mathrm{IV}} + 15u'u''' + 10u''^2 - 20u^3u'' - 45u^2u'^2 + u^6)Q.
\end{aligned}$$

In the equilibrium position, $x = y = 0$, $\phi = \psi = 0$, so that P reduces to 0 and Q to h. Hence, taking *initial values only*,

$$\begin{aligned}
z' &= 0 \ ; \quad z'' = u\cos\alpha - hu^2 \ ; \\
z''' &= 2u'\cos\alpha - u(u+m)\sin\alpha - 3huu' \ ; \\
z^{\mathrm{IV}} &= \{3u'' - u(u^2+um+m^2)\}\cos\alpha \\
&\qquad - (5uu' + 3mu' + um')\sin\alpha - h(4uu'' + 3u'^2 - u^4).^*
\end{aligned}$$

Since $u > 0$, the position of equilibrium is stable if $hu < \cos\alpha$, unstable if $hu > \cos\alpha$, and the first critical case arises when $hu = \cos\alpha$. If this last condition is satisfied, z''' and z^{IV} reduce to the forms :

$$\begin{aligned}
z''' &= -u'\cos\alpha - u(u+m)\sin\alpha \ ; \\
z^{\mathrm{IV}} &= -\{u'' + um(u+m)\}\cos\alpha - (5uu' + 3mu' + um')\sin\alpha - 3hu'^2.
\end{aligned}$$

* Note that u and m are now initial values.

Thus a further condition for stability in this critical case is $z'''=0$, and then

$$z^{\mathrm{IV}} = -\{u''+um(u+m)\}\cos\alpha - u(2u'+m')\sin\alpha$$
$$= -\cos\alpha[u''+u\{m(u+m)-(u'-m')\}\tan\alpha - 3u(u+m)\tan^2\alpha].*$$

The equilibrium is then stable or unstable according as z^{IV} is positive or negative, and a second critical case is given when $z^{\mathrm{IV}}=0$. This may be investigated by considering the initial values of z^{V} and z^{VI} given above.

5. It is obvious that case (ii) is obtained from (iii) by putting $\alpha=0$, that the first critical case arises when

$$hu=1, \quad \text{...(1)}$$

and that a second critical stage is given when this condition is satisfied together with

$$u'=0; \quad \text{...(2)}$$
$$u''+um(u+m)=0. \quad \text{...(3)}$$

Minchin points out that if (1) is satisfied, (2) may be satisfied if A and C are points of maximum or minimum curvature for the upper and lower curves respectively, and then obtains a condition equivalent to (3). It is clear, however, that (1) and (2) may be satisfied in many ways besides this particular one, for $u'=k'+m'=0$ need not imply $k'=0$, $m'=0$. In an example† Minchin states that if (1), (2) and (3) are all satisfied, *i.e.* if $z_0^{(r)}=0$, $r=1, 2, 3, 4$, the equilibrium is stable or unstable according as z_0^{V} is positive or negative. This is the "one-sided stability" referred to above, and is actually a case of unstable equilibrium, for if the body returned to its initial position of equilibrium from a displacement to this side, its kinetic energy in this critical position would carry it to the opposite side, from which it could not return unless $z_0^{\mathrm{V}}=0$. The conditions for stability in the second critical case are thus $z_0^{\mathrm{V}}=0$, $z_0^{\mathrm{VI}}>0$.

Case (i) may be deduced from the general case (iii) by putting $\alpha=0$, $m=0$, so that $u=k=1/\rho$, but, as shown in § 2, the results are simplified so much by the use of ψ instead of s as the independent variable and by the use of the general theorem that ρ must be a minimum, that to obtain (i) from (iii) is of no interest other than as an illustration of the generality of the latter.

In addition to the references to Routh and Minchin given above, the following deal with the subject of this article:

1. J. Larmor, "On Critical or 'Apparently Neutral' Equilibrium" (1883), *Collected Works*, Vol. I, p. 2.

2. A. H. Curtis, *Quarterly Journal*, IX, p. 41.

3. E. J. Routh, *Quarterly Journal*, XI, p. 102.

4. E. J. Routh, *Elementary Rigid Dynamics*, 8th ed. (1913), pp. 364–367; 402–408; 417–422.

I wish to thank Professor H. T. H. Piaggio for the interest he has taken in this article since it was first planned several years ago, for checking all results, and for an improvement in notation which effects a considerable saving in writing the comparatively heavy results in the general case (iii).

F. UNDERWOOD.

1459. During the previous night R.A.F. Bisley bombers started large fires at Gafsa and Sbeitla. One Bisley gunner reported seeing a large explosion fifty miles from the target.—*The Times*. [Per Inst. Lt. F. J. North, R.N.]

* These include all the results given in Routh's *Statics*, p. 179.

† *Ibid.*, 4th ed., p. 139; or 5th ed., p. 91, Ex. 5.

A FRESH APPROACH TO THE MATHEMATICAL CURRICULUM IN SCHOOLS.

BY C. T. LEAR CATON.

(being the Presidential Address to the Midland Branch, given on 4th November, 1944.)

THE urgent necessity for examining thoroughly the content of the Mathematical Curriculum in all types of schools at the present time is my reason for introducing a subject which is becoming a well-worn one, and I hope that in the course of the afternoon I may be successful in showing you what I believe to be a fresh and more fundamental approach to the problems involved, and that by being provocative I may invite a discussion on a very important question.

In the course of my teaching career I have been fortunate in having had the opportunity of viewing mathematical education from varied standpoints, and although now responsible for the administration of a fair-sized school, I still spend a good part of the week teaching mathematics, partly to get to know the pupils, but partly also because I enjoy doing it ; and I must confess that the longer I go on teaching the usual kind of mathematics that goes to form a School Certificate course, the more I am convinced that what I am teaching the boys and girls is really of very little use to them. To ease my conscience, I have spent some spare moments recently looking at a few of the various books and articles wherein my mathematical colleagues, past and present, have given their views on the value of their subject as a staple item of an adolescent's diet. I find there an increasing doubt as to the value of much of the fare offered. Thus, in algebra, one well-known teacher states that on leaving out the inessential parts of the subject, the remainder will be insufficient to justify the retention of algebra as a subject in an examination of School Certificate standard. As regards geometry, there is general agreement that a considerable reduction of material can be made without loss to a pupil's education. Outside the mathematical profession, it has been stated repeatedly, for example by the Committee presided over by Sir Will Spens, that the time given to mathematics in secondary schools could be substantially reduced. Amongst the general public, the significance of the subject is little understood, and parents often seek a knowledge of mathematics for their children merely because the subject is thought to confer a kind of supernatural power, giving the mathematician an intellectual advantage over his fellows.

These facts are symptoms that the existing mathematical education provided in schools is failing to meet present-day needs, and that some fundamental changes are overdue. The familiar proposals for modifying the syllabus do not in my opinion go deep enough. The method of pruning away portions of existing schemes, and grafting in in their place other supposedly more useful topics, results in the production of schedules of slightly differing content but of the same basic character as the original ones. Usually in these attempts, only slight attention is paid to the underlying principles of mathematical education, and to the fact that we have in our schools to provide for widely differing types of pupils. The curriculum of secondary schools taken as a whole has been much overcrowded for a long time ; the impending integration of forms of post-primary education will result in more insistent demands for the inclusion and emphasis of subjects other than mathematics, and the pressure of the curriculum is bound to increase. If mathematics is to retain its traditional place, a convincing case for its value as a part of education for life must be put forward. Many of us would have difficulty in justifying to

an intelligent School Certificate candidate the value of his mathematical knowledge, especially if he were leaving to take a job in which mathematics was not required. Eighty per cent. of our secondary school pupils leave at School Certificate level; their mathematical education is exemplified by the questions set on the usual School Certificate algebra and geometry papers—algebraic manipulations and equations which lead nowhere, absurd problems, geometrical theorems and riders devoid of any connection with reality. Is this sort of thing going to help our pupils along in life? What justification can we put forward for the inclusion of such material?

In my remarks this afternoon, I propose to deal first with the educational principles underlying the inclusion of mathematics in the school curriculum, next to give my views as to why the present syllabus fails to meet the requirements of these principles, and finally to give suggestions for a more valuable type of syllabus. Instead of considering how the present allotment of teaching time can be filled with a varied, but still ill-assorted, collection of mathematical junk, I shall try to classify the types of pupils with whom we have to deal, and then endeavour to decide for each type the minimum mathematical equipment with which they should be provided. If on this basis we can build up well-conceived and well-planned schemes of work, mathematics can continue to enjoy with confidence an assured leading place in the new education.

I shall consider our pupils as divided into the following five categories: (i) future mathematical specialists of university rank; (ii) pupils with definite mathematical ability, who are likely to take up careers in which mathematics is required, including technical students, physicists, accountants, and so on, and others with a marked mathematical interest; (iii) pupils of good intelligence (*i.e.* capable of taking successfully an examination of School Certificate standard), but not especially interested in mathematics, who require some mathematical knowledge as part of a general education; (iv) pupils with low mathematical ability, who must have enough mathematical knowledge to get along in life (some of these pupils may have considerable aptitude in other subjects and may attend Secondary Grammar Schools); (v) pupils of low intelligence, who can only manage to acquire some rudimentary skill at arithmetical computations. This grouping will in itself provide for differentiation between the sexes, relatively few girls coming into the first two categories, and the majority of secondary school girls probably coming in the third group. Differentiation of syllabuses between the groups there must be, as one of our chief failings in grammar schools has been to offer only one kind of course and treatment in mathematics for all types of pupils.

Now in considering briefly the underlying principles of mathematical education, I wish to draw your attention at the outset to what has been called the "inner" and "outer" aspects of the subject. This point has been admirably described by Sir Percy Nunn in *The Teaching of Algebra*: "Mathematical truths always have two sides or aspects. With one they face and have contact with the world of outer realities lying in time and space. With the other they face and have relations with one another. Thus the fact that equiangular triangles have proportional sides enables me to determine by drawing or calculation the height of an inaccessible mountain peak twenty miles away. This is the first or outer aspect of that particular mathematical truth. On the other hand I can deduce the truth itself with complete certainty from the assumed properties of congruent triangles. This is its second or inner aspect." No doubt referring particularly to grammar school pupils, Sir Percy goes on to say: "Our purpose in teaching mathematics in school should be to realise, at least in an elementary way, this twofold significance of mathematical progress. A person, to be really 'educated', should have

been taught the importance of mathematics as an instrument of material conquests and of social organisation, and should be able to appreciate the value and significance of an ordered system of mathematical ideas."

The Association's report, revised in 1928, on *The Teaching of Mathematics in Public and Secondary Schools*, subdivides the first or "outer" aspect into a "utility" aspect and an "outlook" aspect. After enumerating some of the contributions that science has made to civilisation, and pointing out that mathematics lies at the root of the sciences, the report goes on to state: "We assume that the majority of pupils will not make any extensive or profound use of mathematics in after-life, that they will not even be able to follow in detail the mathematical methods of engineering and applied science. But they can be taught so that a vista is opened up to them through which they can see the tremendous potentialities of the study whose elements they are mastering. They should be brought to the stage from which a broad general view may be obtained over the country of applied mathematics; and they should be shown the beginnings of a few of the roads which lead through this country. A public must be created able to realise what science and mathematics are doing for the world, and to form some general conception of the means employed."

Now if such is to be the objective of our mathematical teaching from the "outlook" point of view, how far is this being reached in our schools? For the eighty per cent. of our boys and girls who leave school at the School Certificate stage does the present course, as measured by the questions appearing on their examination papers, give any idea at all of the way in which mathematics lies at the root of such material conquests? Of course, a few portions of the syllabus have an "outlook" value—unfortunately, these usually occur in the optional parts of the syllabus, as for instance numerical trigonometry—but in the ordinary run of school algebra and geometry, the pupil has little chance of forming an impression of the way in which his mathematics has helped the world along. Can he be blamed for thinking that the theorems he has learnt and the riders he has tried to do, the juggling with algebraic fractions and the solving of fantastic problems, the calculation, for example, of the area common to two pennies with centres placed one inch apart—that all this is useless mental lumber of no avail to him personally in life (unless he be set upon a mathematical career), and the use of which to mankind, if any, is a sealed book to him? Can we not sympathise with many of our pupils for finding more value and interest in the study of other subjects? Can we not understand the viewpoint of School Certificate Examining Boards which have decided to remove mathematics from the list of essential subjects for a Certificate?

If the "outlook" value of the work usually done is so slight, is the subject-matter of school algebra and geometry worthy of retention for the sake of its "inner" aspect? It is obvious that a knowledge of the basic concepts of algebra and geometry, for example the use of letters and symbols, acquaintance with planes, lines, points, angles, is a necessity for pupils in all types of schools, but the justification for the inclusion of the usual structure of theorems and riders must depend on the value of its "inner" aspect, especially as the "outward" aspect really receives but little of our attention in teaching. It is the "inner" aspect, the way in which geometrical truths face one another, on which we concentrate. The traditional grounds for including this in the curriculum have been ably summarised by Mr. G. St. L. Carson in an essay, *The Educational Value of Geometry*, published in 1912. Mr. Carson points out that geometry is unique among the sciences in that its basis is to be found in facts agreed by common experience, as opposed to facts derived from experiment, and he goes on to assert that the subject supplies a useful

training in the art of deduction, the full sequence of processes being as follows: (1) the separation of essential from irrelevant considerations involved in the appreciation of points, lines, planes and their mutual relations. (2) The erection on this appreciation of continuous chains of reasoning, one result leading to another in such a way that each chain can be comprehended as a whole and its construction realised as fully as that of each separate link. (3) A discussion of the interdependence of the various premisses and their precise statement. Mr. Carson adds, " Since some such sequence is common to every form of human construction, the educational value of a subject which provides a training in these processes is indisputable."

Now I feel that I must challenge this last statement. Does the geometrical training give in fact the benefits claimed in other forms of human construction? Does the reasoning of the pupils in other fields correspondingly improve? To these questions I think we must give a plain " No " for an answer. The truth is that geometrical material differs from real life situations in that it is highly simplified—in fact, a part of the training of an applied mathematician consists in giving practice in simplifying data so as to make them susceptible to mathematical analysis. The purpose of Euclid's geometry was to set up a particular kind of logical structure, the aim being to reduce the number of postulates to a minimum and to introduce each at as late a stage as possible. The material was deliberately so ordered and arranged as to exhibit the required type of structure. Situations in life are totally different; the material cannot be selected and the situations ordered and analysed like geometry. A variety of other factors enters into consideration, and the logical aspect—usually of minor importance—cannot be isolated. It is well known that those who have had a long and specialised mathematical training are frequently quite unable to deal competently either with a practical situation depending on successful argument, or even with the type of reasoning required in other branches of intellectual activity. The value of logical training in education has been over-rated—it has limited practical usefulness, it does not lead to productive thinking, and it is ill-suited to the personality of boys and girls of school age. Some secondary school pupils may derive benefit from the study of geometry in the later years of their course, when their reasoning powers develop, but the present prominence given to the subject cannot be justified.

The position of geometry is further weakened by two other considerations. In the first place the modern pedagogical tendency to base the subject on a wide series of intuitions negatives the very principle for which Euclid's geometry was set up. Success in school geometry can be assured by a memory well stocked with geometrical facts, plus the ability to appreciate very simple and short chains of reasoning, and by a painstaking learning of the theorems. The new geometry syllabuses being produced do not in any way alter this. These facts in themselves are not an important piece of knowledge, and the time spent in acquiring them is largely wasted except for future professional mathematicians. The art of reasoning could be practised on data of greater intrinsic value. The retention of the usual proofs of the theorems, many of which can be established in a line or two by trigonometry, for example, is no more than the holding on to remnants of a structure which in effect has long been abandoned. Secondly, Euclid's structure itself has revealed cracks under the searching analysis of modern mathematicians; the structure does not rest entirely on the premisses, but also depends on observation in a subtle way. Unfortunately for the logical training, this is evident to our pupils. No doubt most teachers are familiar with the fallacy in Mr. Rouse Ball's *Mathematical Recreations and Essays* wherein it is proved that a right angle is equal to an angle greater than a right angle. Each step is a correct link

in a chain of reasoning usual in riders, and yet the result is false. Some factor other than logic is involved. If an accurate diagram is drawn, observation will supply the missing factor. As in life, the correctness of the conclusion depends on accurate observation as much as on sound reasoning. The schoolgirl put her finger right on the spot when she stated, "I can prove the triangles congruent, but they aren't." In other words, the pupil's intuitive distrust of geometrical arguments which "prove the obvious" is well-based. He is right when he thinks the evidence of his senses superior to a geometrical demonstration, and considers the latter a mere tissue of words. We are wrong in so far as our teaching induces him to rely on reasoned arguments, and such training can be destructive of correct, as well as of productive, thinking.

The educational benefits claimed for geometrical training disappear on analysis. The dethronement of Euclid has left behind a hollow structure. The subject, apart from a core which needs to be retained, does little if anything to show how mathematics affects the progress of civilisation and how it lies at the root of the sciences, and its internal value is very limited. In other words, our pupils learn nothing very much from the methods, and the results when they get them are of no use. The case for the retention of the bulk of the algebra course is even weaker. Its outward-facing aspect cannot be appreciated within the limits of the school course. Most of the work is a training in technique for the future mathematician or technical student, and is devoid of meaning and aim for pupils who leave at School Certificate stage to enter non-mathematical occupations. In its inner aspect, algebra as taught is devoid of structure, being a collection of miscellaneous methods of manipulation having no general educational value. I suggest then that the method of patching the present syllabuses in geometry and algebra cannot possibly meet the situation, and that it is better to throw overboard completely algebra and geometry in their existing forms and to consider afresh what parts of mathematics are of value for our five categories of pupils.

Secondary Grammar Schools will contain pupils in all of grades (1) to (4); those in category (5) are not likely to enter such schools, but will be allocated to "modern schools" where they will be given a course in arithmetic, treated from the strictly utilitarian viewpoint. Children in categories (1) to (4) will be distributed over several types of secondary schools, and all these pupils should receive, in addition to their arithmetic training, a course in the basic concepts of algebra and geometry. This course would include the mathematics necessary for leading the life of an ordinary citizen in the modern world, and an attempt would be made to give the work as much "outlook" value as possible. We ought to include, for example, the use of symbolism in algebra, the means of substituting in a formula, an idea of directed number, some treatment of graphs, an acquaintance with mathematical devices such as logarithms and the slide rule, and a training in the use of mathematical tables. Some simple equations and the "I think of a number . . ." type of problem would be worth doing (at the least, these enable the pupils to get a lot of fun out of the puzzles in the newspapers!), but not much in simultaneous or quadratic equations or factors. The ideas of a rate of change and an integral would have to come in for all pupils. In geometry, an acquaintance with concepts, three-dimensional as well as two-dimensional, would be made, that is to say, planes, angles, lines, and so on; practice in the use of mathematical instruments would be given; the idea of similarity would be introduced—in fact, all that comes into a good "Stage A" course, plus, perhaps, one or two high-lights such as Pythagoras' and Pascal's theorems treated in some way—little or no formal proofs would be given, but demonstrations (for example, of the angle sum of a triangle) would be appropriate to the course. Congruence (one of the chief methods of proving the obvious) is an

instance of what would be omitted. Finally, elementary trigonometry would have a place in the scheme.

These subjects of algebra, geometry, trigonometry, and so on, must be completely fused—they must not be dealt with in consecutive chapters in the textbooks—practice material would be included, numerical examples in geometry, practical training in drawing and measurement, and so forth. The first aim of the course being to assist a pupil in getting along in life, the methods taught must be those which are simple and practically useful, easily understood and therefore easily remembered; whether they are the best for illustrating the mathematical theory, and the most easily deducible from it, will be of secondary importance; examples worked by the pupils will be chosen to give clear illustrations of principles and method, "artificial" rather than "practical" examples being quite acceptable, provided they are clear, interesting, and show up the methods without confusing complications. Subsequently, examples showing the applications of the mathematics to practical life will be included in order to achieve the second aim of showing forth the way in which mathematics has helped to advance civilisation and the sciences. The "outlook" aspect will influence throughout the whole choice and treatment of topics to be included in the course. In addition, the minds of our teachers must be well stocked with information regarding mathematical history, so that their lessons can be enlivened with a broader interest, and the contribution that mathematics has made in the past can be brought out incidentally in the teaching.

In this way an interesting syllabus could be worked out which would give at the same time a practical and a cultural training. With a suitable approach it could be made to appeal to and be assimilated by girls and boys of ages 11 to 14 years in all types of post-primary schools. In Secondary Grammar Schools most of the work could be covered in four or five periods a week in the first three years, the rest being taken in a smaller number of periods in the remaining years, and the subject being examinable in School Certificate under some such title as "Elementary Mathematics I", to be taken by almost all candidates. The course would provide essential intellectual equipment for every intelligent citizen of the modern world.

We have next to consider what additional mathematical work should be taught to boys and girls in category (3), that is, pupils at Secondary Grammar Schools who are not particularly mathematical, but who need to have some mathematical training as part of a general education. Why is mathematics regarded as an essential part of a liberal education for our more intelligent children? The answer to this question has been well expressed by Sir Percy Nunn in his book *Education; its Data and First Principles*: "A subject such as Mathematics represents a tradition of intellectual activity that has for centuries been directed towards a special class of objects and problems. In generation after generation, men, sometimes of outstanding genius, have studied these objects and worked at these problems; accepting, correcting, expanding the methods and knowledge of their predecessors, and handing on the results of their own labours to be treated in the same way. Thus has grown up a distinctive type of intellectual activity, exhibiting a well-marked individuality, and informed by a characteristic spirit. . . ." Now it is the special function of the Secondary Grammar School to introduce its pupils to these well-marked types of intellectual activity. Here we have to deal with mathematics in its inner aspect, to consider its structure—the way in which its truths face one another. This is the significance of the sequence of processes described for the particular case of geometry; the selection and clarification of concepts and premisses, the erection of continuous chains of reasoning. The characteristic activity of mathematics is the ordering of

material—the setting up of a structure. The difficulty is to select material to show this characteristic activity which is within the scope of boys and girls aged 15 and 16 years. As A. N. Whitehead says in *An Introduction to Mathematics*: "The study of mathematics is apt to commence in disappointment. The important applications, the theoretical interest of the ideas, the logical rigour of its methods, all generate the expectation of a speedy introduction to processes of interest. Yet, like the ghost of Hamlet's father, this great science eludes the efforts of our mental weapons to grasp it. The reason for the failure of mathematics to live up to its reputation is that its fundamental ideas are not explained to the student disentangled from the technical procedure which has been invented to facilitate their exact presentation in particular instances. The unfortunate learner finds himself struggling to acquire a knowledge of a mass of details not illuminated by any general conception." It is this general conception that somehow we have to bring to the fore. The traditional attempt has been made through the subject-matter of Euclidean geometry, but I have already explained the grounds for my opinion that this may not be the best material for the purpose, even if its study is deferred until an age when powers of reasoning begin to develop. Euclid's geometry is 2000 years old, and it is likely *a priori* that in this interval other more suitable branches of mathematics will have sprung up. Further, the study of geometry or any one branch of mathematics is altogether too narrow a field to show any general conception. In addition, the study of geometry does not make much contact with the more recent stretches of mathematical developments. The mathematical country is now a very large one, and without losing sight of our primary aim to exhibit the characteristic activity of mathematics we want, if we can, also to lead our pupils along several roads into the territory, particularly towards the most interesting and profitable, and the more newly explored portions of the country, and to give them a bird's-eye view of the whole.

I suggest then that instead of limiting the work to geometry we make a list of several topics, each of which illustrates the characteristic mathematical activity. Several of these, possibly not all, will be studied in each school. Some of the topics must be studied in some detail and with thoroughness, the necessary degree of technical facility first being acquired, so as to give sufficient depth to the work and to enable the vital nature of the mathematical thought to be properly apprehended and assimilated. At the same time, attention must not be confined to technical detail, the general conception must be kept steadily in view. The topics must also be pursued far enough to show where they are leading; we must not repeat the mistake of the present algebra syllabus, and provide something the ultimate value of which will only be seen by those who stay on at school for advanced mathematical courses. Any way in which the topics make contact with the external world, now or in the past, will naturally be explained by the teacher.

By way of illustration, I think that suitable approaches could be worked out for the following topics, to be taught to pupils in their last two years before taking the School Certificate examination and after completing most of the basic course already outlined:

(*a*) All boys and girls are very interested in the fact that it is possible to calculate π to 100 places of decimals, and wonder by what method logarithms and cosines can be determined to any desired degree of accuracy. Infinite Series is a branch of mathematics of vast importance; it is well adapted to exhibit the way in which mathematicians typically handle their material; it has an obvious and wide "outlook" value.

(*b*) Generalisation of Number is another topic of great mathematical importance. A course could be presented showing the successive introduc-

tion and use of irrational numbers, real numbers, algebraic numbers, complex numbers. Again we have the advantages of appeal to the pupils, practical value, and illustration of typical mathematical method.

(*c*) The Nature of Coordinate Geometry; dealing with curves other than circles in two and three dimensions; abstract geometry; the principle of continuity; position and loci in two and three and more dimensions; tracks on the Earth's surface and the motions of the stars.

(*d*) Pure Geometry, but with a different content and approach. Euclid's geometry is far surpassed in beauty and power by modern descriptive and projective geometry. Could not this be brought within the range of pre-School Certificate candidates? Non-Euclidean geometry is another possible topic.

(*e*) Probability and Statistics. There is scope here for the introduction of a course to show how collections of random data have been reduced to order by the methods of mathematical analysis. Here again there is plenty of interest for the pupils, while the teacher can bring out the "outlook" value of the subject, at the same time showing the characteristic mathematical treatment.

(*f*) The idea of Infinity—a subject of absorbing interest. A short sequence of ideas could be worked out on the taming of the idea of Infinity; the arithmetical, algebraic and geometrical aspects could be brought in, also the idea of a Line at Infinity; possibly Cantor's work on transfinite numbers could be introduced in a simple way.

(*g*) Finally, there is the whole domain of Applied Mathematics, by which I mean much more than Mechanics. The teaching of Mechanics is the task of the Physics master, but when he has dealt with the experimental side of statics, dynamics, electricity and magnetism, light and sound, it is for the mathematics teachers to show the effect of the application of mathematical analysis to some of these subjects, thus securing an excellent opportunity for illustrating the characteristic mathematical method.

Your first reaction to these suggestions may be that the topics are too difficult for pupils below School Certificate standard, but then we must remember that until the "Association for the Improvement of Geometrical Teaching" came into being, geometry was a subject scarcely understood by teacher or pupil. If the teaching of these topics is worth while, then suitable methods of approach will render the material as capable of assimilation as in the case of geometry. The approach must be quite different from that of the previous basic course, for we are now focusing attention on the inner relations of the ideas so that the treatment needs to be mathematical and abstract. Pupils aged 15 and 16 years can be made to appreciate abstract thought—if the contrary, we must clearly give up all attempts to teach the nature of mathematical activity in schools and leave this to the universities. This work would form an examination subject, "Elementary Mathematics II", not to be taken by pupils with low mathematical aptitude, but to be taken by most Secondary Grammar School pupils. A time allowance of two or more periods a week during the last two years of the course, with one topic covering each term, would suffice.

Finally, I must refer briefly to categories (1) and (2). The course just outlined will be equally suitable for the mathematical specialist and the technical student. It will give the former a broad idea of the method of mathematics disentangled from technical detail, and for the latter it will provide the element of a liberal education as regards mathematics which I suspect is lacking in many technical courses. Pupils in the first two grades will need more time for mathematics during the last two or three years before School Certificate, and should prepare for additional subjects which we might ca-

"Additional Mathematics" for the Secondary Grammar Schools, the Technical High School students no doubt having a suitable examination of their own. This course in Additional Mathematics will be essentially vocational, and will aim at giving the future mathematician or accountant or physicist what he needs for the further development of his subject. We must provide for the requirements of science students at School Certificate level, for pupils who will leave at this stage to take up careers involving mathematical work, as well as giving a technique for those who wish to follow mathematics or physics courses in the Sixth Form. Surely this is the place to insert the manipulative work in algebra. I think the Pure Geometry could mostly be left to the Sixth Form; it would be more useful to include elementary coordinate geometry and calculus and some algebraic trigonometry in the "Advanced Mathematics" syllabus. A mathematical treatment of some branches of applied mathematics must also be given. These subjects could all be treated without too great stress on either rigour or memory work—for example, proofs of formulae in trigonometry need not be learnt—the aim being to acquire a reasonable degree of technical facility in as broad a field as possible. This would leave the specialist mathematician in category (1) with a good foundation on which to base a study of mathematics with full rigour in the Sixth Form where this properly belongs.

Sixth Form work leads through Entrance Scholarship and Higher School Certificate Examinations to the Universities, and in consequence the content of Sixth Form mathematics will largely be determined by University requirements. Very good suggestions have been made recently for mathematics courses for Physics students and scientists in the Sixth Form, and I do not propose in this paper to deal with such work. The urgent problems are concerned with the pupils below Sixth Form level. In 1871 this Association succeeded in awakening interest in the presentation of Euclidean Geometry, which resulted in a profound improvement in the teaching of mathematics generally in schools. The time has come for it to take the lead again, to throw off the shackles of the past, and to replan the mathematics courses in all types of schools to fit in with the new educational structure and to contribute more effectively to the needs of the post-war world. In this process I hope our Branch will be able to play its full part in originating, in stimulating, in inspiring. In a short address I have been unable to do more than throw out a few suggestions, but if there is anything in what I have said which will contribute to this goal, I shall be amply repaid for the task of preparing this paper.

C. T. L. C.

1460. I shared from the first the aversion which nearly all the most intelligent men I have met since have had for science, and most of them for mathematics. . . . It is true that I was very badly and unsympathetically taught in both mathematics and science. Not bad at arithmetic and algebra, I had no aptitude for or interest in geometry. I had no logical faculty: or rather, it was never aroused at school; all that came later with maturity and the endless discussions of undergraduate life at Oxford.—A. L. Rowse, *A Cornish Childhood* (Cape, 1942), p. 174. [Per Mr. F. W. Kellaway.]

1461. "She had had to do a chest tap. Chest taps . . . were almost as common as spinals. But she hadn't been able to stop thinking of the one person in a thousand who, for some mysterious reason nobody understood, went out like a light during a chest tap. It was an idiotic thing to worry about, one in a thousand *or less than a thousand*. . . ."—Hannah Lees, *Women will be Doctors*, 1942, p. 90. Our italics. [Per Professor E. H. Neville.]

MATHEMATICAL NOTES.

1827. *A geometrical construction for the triangle in Note* 1740.

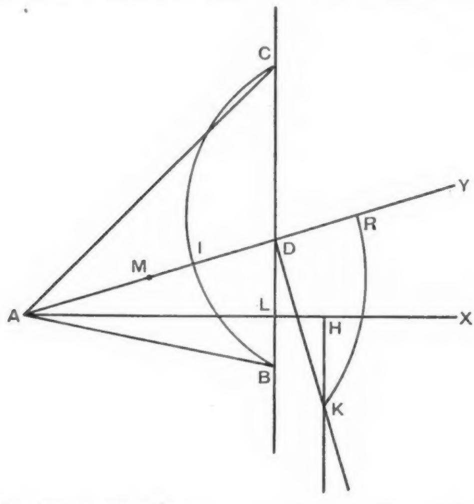
Let the bisector of the angle A meet BC in D, and let I and I_1 on AD and AD produced be incentre and excentre. Then it is known that :

(i) the perpendicular AL on BC is inclined to AD at an angle $\frac{1}{2}(B-C)$; thus AD is known.

(ii) the projection of II_1 on $BC = b - c$; thus II_1 is known.

(iii) IBI_1C is a cyclic quadrilateral, the centre of the circle being the mid-point of II_1.

(iv) A and D are harmonically conjugate with I and I_1, and therefore the circles on AD and II_1 as diameters are orthogonal.



Hence the following construction :

On a straight line AX set off $AL = p$, $LH = \frac{1}{2}(b-c)$. Draw AY so that $\angle XAY = \frac{1}{2}(B-C)$. Through L draw a line perpendicular to AL cutting AY in D. Draw DK perpendicular to AY and HK perpendicular to AX. With centre M, the mid-point of AD, and radius MK describe an arc cutting AY in R. Then a circle, centre R, and radius equal to KD will cut the line through L and D in B and C.

The analytical result given in Note 1740 can be simplified by introducing a subsidiary angle ϕ defined by the equation $\tan \phi = (d/p) \cot \theta$. It will then be found that $\sin \frac{1}{2}A = \cos \theta \tan \frac{1}{2}\phi$. S. G. HORSLEY

1828. *Note on Dr. E. A. Maxwell's article* (*Gazette*, XXVIII, May, 1944, p. 51).

1. A good deal more can be got out of Dr. Maxwell's article on " A Doubly Infinite System of Cyclic Quadrilaterals ", in the May *Gazette*. It is proposed here to make easy deductions and to call attention to some difficult ones for others to tackle. The present writer does not wish to keep all the fun to himself.

In the middle of p. 51, the equation

$$\{(a-b)S-X^2+Y^2\}/(hS-XY)=\mu$$

shows by inspection that Σ passes through the foci of $S=0$, since their equations are $S=(X^2-Y^2)/(a-b)=XY/h$.

Dr. Maxwell makes no mention of the case when $S=0$ is a circle. Here $a=b$, $h=0$, and so P_1P_2 subtends a right angle at its centre. We can thus take the joins of P_1 and P_2 to the centre as axes of coordinates and find the equation of the circumcircle of the quadrilateral of tangents from P_1 and P_2 in a simple form.

Let P_1 be $(x_1, 0)$, P_2 be $(0, y_1)$, and the circle be $x^2+y^2-a^2=0$.

The equation of the pair of tangents from $(x_1, 0)$ is

$$(x^2+y^2-a^2)(x_1{}^2-a^2)-(xx_1-a^2)^2=0,$$

or

$$-a^2x^2+(x_1{}^2-a^2)y^2+2a^2x_1x-a^2x_1{}^2=0.$$

Similarly the pair of tangents from $(0, y_1)$ is

$$(y_1{}^2-a^2)x^2-a^2y^2+2a^2y_1y-a^2y_1{}^2=0.$$

Multiply the first equation by $y_1{}^2$, the second one by $x_1{}^2$ and add the results. Then the required circle is

$$(x_1{}^2y_1{}^2-a^2x_1{}^2-a^2y_1{}^2)(x^2+y^2)+2a^2x_1y_1(xy_1+yx_1-x_1y_1)=0,$$

or

$$\left(\frac{1}{a^2}-\frac{1}{x_1{}^2}-\frac{1}{y_1{}^2}\right)(x^2+y^2)+2\left(\frac{x}{x_1}+\frac{y}{y_1}-1\right)=0,$$

and the radical axis of this circle and $x^2+y^2-a^2=0$ is parallel to P_1P_2. When P_1P_2 passes through a fixed point (x_2, y_2), so that $x_2/x_1+y_2/y_1-1=0$, these circles will presumably have an envelope, the discovery of which is left to others.

2. We can also easily find the corresponding circumcircle for $y^2-4ax=0$, and it is more convenient to take the focal chord on which P_1 and P_2 lie as $x=a+my$. The equation of the tangents from (x_1, y_1) is

$$(y_1{}^2-4ax_1)(y^2-4ax)-\{y_1y-2a(x+x_1)\}^2=0,$$

or

$$x_1y^2-y_1xy+ax^2+(y_1{}^2-2ax_1)x-x_1y_1y+ax_1{}^2=0.$$

Similarly for those from (x_2, y_2). Multiplying the first equation by y_2, the second by y_1, subtracting and putting $x_1=a+my_1$, $x_2=a+my_2$, we obtain

$$a\{(x-a)^2+y^2\}=y_1y_2(x-my+am^2).$$

Hence if P_1 or P_2 is fixed, and the other varies along a fixed focal chord through P_1 or P_2, these circles form a coaxal system with real limiting points, viz.: the focus $(a, 0)$ and the point $(-a, 2am)$; the radical axis being parallel to P_1P_2.

3. The same problem for the interchangeable rectangular hyperbolas of § 2·3 in the article must be very difficult to solve, but there is probably an interesting set of circles common to both.

N. M. GIBBINS.

1829. *Further notes on elliptic function theory.*

In an article in the *Gazette* for May, 1944, it was shown that the theory of Jacobian Elliptic Functions can be developed in a simple, direct manner, without using sigma or theta functions. No proof was, however, given of the fundamental relation

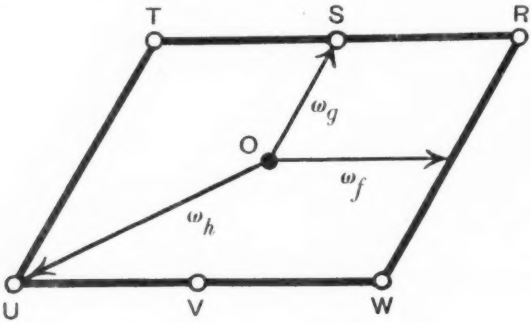
$$gj\,\omega_h=v\,hj\,\omega_g \quad\text{.......................................(1)}$$

in Neville's notation ($cs\,\omega_d = v\,ds\,\omega_c$ in the notation of the article). v is here defined to be i or $-i$ according as ω_f, ω_g, ω_h come in a positive [anticlockwise] or negative [clockwise] order around the origin in the Argand diagram.

This may, however, be proved by a simple contour integration. Consider the parallelogram $RSTUVW$, where

$$R = -U = -\omega_h, \quad S = -V = \omega_g, \quad T = -W = \omega_g - \omega_f.$$

This parallelogram, with the sense $RSTUVWR$, surrounds the origin once in a positive or negative sense according as $v = i$ or $-i$ respectively (Fig. 1).



Thus, since $fj\,z$ has a pole of residue 1 at 0, and no poles elsewhere in the parallelogram,

$$\int_{RSTUVWR} fj\,z\,dz = 2\pi v. \quad \text{.................................(2)}$$

Now by the relation $fj(z+2\omega_f) = fj\,z$ the parts of this integral along the sides TU, WR sum to zero; while by the relations $fj(z+2\omega_g) = -fj\,z = fj(-z)$ the four integrals along RS, ST, UV, VW are all equal, and hence by (2) each is $\frac{1}{2}\pi v$,

$$\int_R^S fj\,z\,dz = \tfrac{1}{2}\pi v. \quad \text{...(3)}$$

But the indefinite integral of $fj\,z$ is $-\ln(gj\,z + hj\,z)$,* as is obvious on differentiation. Substituting in (3) and simplifying by the relations

$$gj\,\omega_g = 0 = hj\,\omega_h,$$

we get

$$\ln gj\,\omega_h - \ln hj\,\omega_g = \tfrac{1}{2}\pi v,$$

which is equivalent to (1).

It seems worth while also noting some simple forms of the addition theorems. Wilkinson † has shown that if $x+y+z=0$, then

$$fj\,x\,gj\,y + gj\,z\,hj\,x + hj\,y\,fj\,z = 0. \quad \text{............................(4)}$$

(For the left-hand side may be seen to be for fixed z, an elliptic function of x without poles, *i.e.* a constant independent of x and y, and, by symmetry, of z. Putting $x = \omega_f$, $y = \omega_h$, $z = \omega_g$, we find its value to be 0.)

* ln = log. nat.

† *Proc. 5th Inter. Cong. Math.* (1912), 1, p. 407.

By permuting x, y and z we get six equations like (4), which may accordingly be considered as *six* linear equations for the *three* unknowns

$$fj\,z = -fj\,(x+y),\ gj\,z,\ hj\,z,$$

which have accordingly a great number of solutions. But these solutions may be considerably simplified in form by bringing in the function

$$F(x) = -d \ln fj\,x/dx = gj\,x \,.\, hj\,x/fj\,x, \quad \text{.......................(5)}$$

and similarly $G(x)$, $H(x)$.

These functions F, G, H are of some interest on their own account: they all have the same primitive periods $2\omega_f$, $2\omega_g$, $2\omega_h$; and

$$GH = (fj\,x)^2,$$
$$F(H-G) = g_h{}^2,$$
$$gj\,x + hj\,x = F(\tfrac{1}{2}x).$$

As "unknowns" in the equations we use $f_0 = fj\,(x+y)/fj\,(x)fj\,(y)$, and g_0, h_0 similarly defined.

Equation (4) then takes the form

$$f_0G(y) + g_0F(x) = 1, \quad \text{.................................(6)}$$

with five other similar equations on permuting x, y and also f, g, h.

Some forms of the solution of these equations are (omitting brackets for clearness):

$$\begin{aligned} f_0 &= (Fx - Fy)/(GxHx - GyHy) \\ &= (1 + f_g{}^2/HxHy)/(Gx + Gy) \\ &= (1 - f_g{}^2f_h{}^2/GxGyHxHy)/(Fx + Fy), \end{aligned}$$

while $fj\,(x+y) = f_0 fj\,x \,.\, fj\,y$. CEDRIC A. B. SMITH.

1830. *Illustration for $x^0 = 1$.*

The average child is probably inclined to view with suspicion this curious result of his study of index laws in the algebra class. Teachers may like to know of a concrete way of illustrating it.

A fairly large sheet of uncreased paper, rectangular in shape, is folded repeatedly so that each fold divides the surface into two equal rectangles. The class is asked to observe that if the sheet is opened out again after any folding operation, it is found to be divided by the creases into a number of parts equal to some power of 2. Thus 1 fold gives 2^1 compartments, 2 folds give 2^2 compartments, and so on; n folds give 2^n compartments. It only remains for the class to realise that for the original unfolded sheet n was equal to zero, and that it contained 1 compartment, for it to be convinced that $2^0 = 1$.

In a similar way, if each folding operation consists of dividing the rectangular surface into 3 equal rectangles, it can be shown that on opening out the sheet after $1, 2, 3, \dots, n$ operations, the sheet is creased into $3^1, 3^2, 3^3, \dots, 3^n$ compartments; and again $n = 0$ corresponds to the original uncreased sheet having a single compartment. Hence $3^0 = 1$.

The extension, on similar lines, to the case x^n (where each operation consists of folding the surface into x equal areas) is a simple matter for the teacher. Here again $n = 0$ corresponds to the unfolded sheet having 1 compartment. W. G. G.

1831. "*A minus and a minus make a plus.*" (See Notes 1622 and 1724.)

There are four distinct cases, each requiring its own treatment.

(i) $a-(b-c)\equiv a-b+c$.

This applies to signless (as well as directed) numbers, and is therefore the first to be met. It is one of a family of arithmetical theorems, viz. :

$$a+(b\pm c)\equiv a+b\pm c,$$
$$a-(b\pm c)\equiv a-b\mp c.$$

To single it out as quainter or less obvious than the others is to ask for trouble. One lesson should convince any normal beginner of the almost obvious truth (for signless numbers) of these theorems : the best method is undoubtedly Mr. Langford's (Note 1622) geometrical illustration followed by numerical examples. Immediate application, to the solution of easily (and independently) checked problems, helps to clinch the matter.

(ii) $a-(-b)\equiv a+b$.

Here again we have one of a family, which should be dealt with *as* a family:

$$a+(\pm b)\equiv a-(\mp b).$$

These apply only to directed numbers. Addition of such numbers should (I suppose) be dealt with by considering the addition of displacements on a line. Subtraction follows, as the answer to the question, "What must be added to y to make x?" A series of numerical examples will then soon make it clear that the subtraction of a directed number is equivalent to the addition of the corresponding number of the opposite sign, and *vice versa*. The analogy between $a-(\pm b)\equiv a+(\mp b)$, which enables us to convert subtraction to addition in directed arithmetic, and $\frac{a}{b}\div\frac{p}{q}\equiv\frac{a}{b}\times\frac{q}{p}$, enabling us to convert division to multiplication in fraction arithmetic, is worth mentioning : but, like all analogies (*e.g.* Mr. Stewart's analogy with the logical double negative: Note 1724 (*d*)), it should appear only after full conviction has been reached, in order that there may be no suggestion that it constitutes *any* sort of proof.

I was myself, when young, imposed upon by the following :

" We know $\qquad a-(b-c)\equiv a-b+c.$

$\therefore$ (putting $b=0$) $\qquad a-(0-c)\equiv a-0+c,$

i.e. $\qquad a-(-c)\equiv a+c.$"

The retailer of this sophistry, who should have been put in prison for leading children astray, was inordinately proud of it : the kindest suggestion one can make is that Nature had denied him the intelligence required to see through it.

(iii) $(-a)\times(-b)\equiv(+ab)$.

It must be made clear from the first that this is a *definition*, not a theorem. $(\pm a)\times(\pm b)$ is, when it first appears, a meaningless combination of symbols, like "*CUG*". We can, if we choose, give it a meaning—any meaning which suits our convenience. The meaning we *do* give it is $(\pm ab)$, according as the signs of the factors are the same or different. It is nonsense to talk of *proving* that $(-1)\times(-1)=(+1)$; all that may legitimately be done is to give the reason why *we have decided* that it shall be so. The reason is, of course, that we wish to have one set of general rules applicable to all our different arithmetics—signless, fractional, directed and (later) complex—not a different set for each. The rule "distance = speed × time" still holds (in a slightly modified form) with directed numbers, if, and only if, we define multiplication in a certain way * : but there is no objection to defining multiplication in some other way, provided we are prepared to work with a different relation between

* I take this excellent example from *Elementary Algebra*, by Siddons and Daltry.

distance, speed and time when directed numbers are involved. Similar remarks apply to such rules as $a-x(b+c)\equiv a-xb-xc$, already known to be true for signless numbers. It is instructive to adopt (temporarily!) some other definition, such as "the sign of the product shall be negative unless both factors are positive", and to show that with this definition a new set of rules for the removal of brackets is necessary. "$\frac{a}{b}\times\frac{p}{q}\equiv\frac{ap}{bq}$" is also a definition: it is easy to show that the definition "$\frac{a}{b}\times\frac{p}{q}\equiv\frac{a+p}{b+q}$" (to which there is no logical objection) would render the rule "area of rectangle = breadth × length" inoperative for fractions, and demand a new one.

The dangerous fallacy lies in inverting these arguments; in assuming that the old rules must of necessity remain true with the new numbers, and "proving" from this assumption that $\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$ or that $(-2)\times(-3)=(+6)$. The arguments (*b*) and (*c*) of Note 1724 both involve this fallacy.

(iv) $-(-a)\equiv+a$.

Here $-(-a)$ is a conventionally shortened form of $(-1)\times(-a)$; and we have a special case of (iii).

It is incorrect to treat these different cases as examples of the operation of a single rule: they exemplify three distinct rules, and the resemblance is only superficial. Confusion must, therefore, result from using some compendious (and barbarous) slogan, such as "two minuses make a plus", to cover them all. In any case the slogan is untrue. In (i), *e.g.*, we have a minus and a minus making a minus and a plus, not just a plus. And: "Please, Sir, a minus and a minus make a plus, so why isn't $-1-2$ equal to $+3$?"

L. W. H. Hull.

1832. *Centre of pressure.*

As a boy at school I used this rule to remember the position of the centre of pressure of a body bounded by straight lines with one point (or line) in the surface of a fluid.

Express the position of its C.G. in its lowest terms and add 1 to the numerator and denominator.

E.g. rectangle C.G. $\frac{1}{2}$ way down,

C.P. $\frac{2}{3}$ way down.

Having not met anyone in the last twenty-five years who knows it, I am writing to enquire if it really is as well known as it ought to be.

The mathematics behind it is interesting. Its limitations are obvious.

D. A. Grenfell.

1833. *On Note* 1737.

I accept Mr. Ramsey's correction requiring the rate of change of momentum to be written as $M\,dV/dt$ rather than $d(MV)/dt$ in my note in the *Gazette* of February, 1944. There appear to be two points of view. Either we write, for example,

$$M\,dV/dt=F-F',\text{ where }F'=m(V-v),$$

or

$$M\,dV/dt+m(V-v)=F,$$

regarding the former equation as representing the motion of the engine, and the latter as the equation of motion of a system consisting of the engine and the matter $m\,.\,\delta t$ about to be attached. It is, perhaps, worth while to note that, considered in this way, the equations under discussion are merely expressions of D'Alembert's principle.

H. Langdon-Davies.

1834. *On Note* 1753 (*Asymptotes*).

The appropriate method of finding the asymptotes of a hyperbola depends upon the point of view. What *is* an asymptote?

(i) Note 1753 regards it as the limit of a tangent. Is there any advantage in the use of $\sec\theta$ and $\tan\theta$? With an ordinary algebraic parameter the tangent is $(1+t^2)x/a - 2ty/b = 1-t^2$ and $t\to +1$, $t\to -1$ give the asymptotes.

(ii) If an asymptote is defined as a tangent at infinity, the curve being $x^2/a^2 - y^2/b^2 = z^2$, then $xx_1/a^2 - yy_1/b^2 = zz_1$ is the tangent at (x_1, y_1, z_1), and this includes the cases of $(a, b, 0)$ and $(a, -b, 0)$. Alternatively, the equation $(x/a+y/b)(x/a-y/b) = z^2$ shows that the curve touches $x/a \pm y/b = 0$ on $z=0$.

(iii) Taking the point of view of indefinitely close approach, suppose that (x, y) is a point on the curve in the first quadrant. If y is sufficiently large, say $y/b > N$, it follows from $(x/a+y/b)(x/a-y/b) = 1$ that $(x/a - y/b) < 1/2N$, and so the point (x, y) is indefinitely close to $x/a = y/b$. A. R.

1835. *On Note* 1758 (*Director Circle*).

The equation $Al^2 + 2Hlm + Bm^2 - 2(Gl+Fm)(lx+my) + C(lx+my)^2 = 0$, found by elimination of n from the envelope equations of the conic and the point (x, y), is the equation of the points at infinity on the tangents from (x, y). Hence the director circle is found by expressing the condition for these points to be in perpendicular directions, namely that the sum of the coefficients of l^2 and m^2 is zero. The foci are found from the same equation by identifying it with the equation $l^2 + m^2 = 0$ of the circular points. A. R.

1836. *A problem in probability.*

Six cricket clubs form an association and compete each year for a premiership. The winning team receives a shield, which is to become the absolute property of the first club to win the premiership three years in succession. What is the minimum number of years that must elapse before a person not knowing the year-to-year results can say that the odds that the shield has been won outright are at least evens, it being assumed that in each year each team has an equal chance of winning the premiership?

Let p_n be the probability that the shield has not been won outright at the end of n years. This will be so if either

(1) The team winning the nth competition did not win the $(n-1)$th, and the shield had not been won outright after the $(n-1)$th competition—chance of this is $\frac{5}{6}\cdot p_{n-1}$.

(2) The team winning the nth competition also won the $(n-1)$th but not the $(n-2)$th, and the shield had not been won outright after the $(n-2)$th competition—chance of this is $\frac{1}{6}\cdot\frac{5}{6}\cdot p_{n-2}$.

Thus
$$p_n = \tfrac{5}{6}\cdot p_{n-1} + \tfrac{1}{6}\cdot\tfrac{5}{6}\cdot p_{n-2}.$$

This is the relation connecting successive coefficients of a recurring series $\sum_{n=1}^{n=\infty} u_n x^{n-1}$ with scale of relation $1 - \frac{5}{6}x - \frac{1}{6}\cdot\frac{5}{6}x^2$, of which the generating function is

$$\frac{1}{2\sqrt{45}}\left\{(\sqrt{45}+7)\cdot\frac{1}{1-\dfrac{\sqrt{45}+5}{12}x} - (7-\sqrt{45})\cdot\frac{1}{1+\dfrac{\sqrt{45}-5}{12}x}\right\}$$

and

$$u_n = \frac{1}{2\sqrt{45}}\left\{(\sqrt{45}+7)\left(\frac{\sqrt{45}+5}{12}\right)^{n-1} - (-1)^{n-1}(7-\sqrt{45})\left(\frac{\sqrt{45}-5}{12}\right)^{n-1}\right\}.$$

$$[u_1 = u_2 = 1].$$

The required value of n is found by equating this expression to $\frac{1}{2}$, the next integer above the value of n so obtained being the required number of years. It is immediately found that the 2nd term is negligible and the solution of the equation then presents no difficulty, the appropriate value of n being 31.

A general solution for m teams is readily obtainable; as a matter of interest, it may be noted that the appropriate values of n for even numbers of teams from 4 to 12 are as follows:

No. of teams.	n.
4	15
6	31
8	51
10	78
12	110

P. M. Wickens.

1837. *Teaching of geometry and statistics.*

For several years greater emphasis has been placed on direct measurement in introducing the principles of elementary geometry. This is an excellent move in principle, though I am not clear whether it has been reconciled with the rejection of the method of superposition, which has been so heartily condemned in the past. Some years ago, in the Mensuration chapter of my *Scientific Inference*, I constructed a Euclidean theory based on experimental laws, and developed it up to the establishment of rectangular coordinates in three dimensions and the Cartesian formula for distance. This was not intended for elementary teaching, and in some respects would probably be unsuitable, since I defined angles and planes in terms of distances and accordingly met some analytical complications. Nevertheless it has one notable advantage over the modern textbooks that I have examined—it avoids the negative definition of parallels. We cannot test to any useful degree of accuracy in practice whether a pair of lines never meet. I should be very glad to know the opinions of teachers on the possibility of adapting my method to their purposes, for instance, by introducing measurement of angles at an earlier stage.

The use of vector methods is sometimes advocated, but some of the theorems often " proved " by vector methods seem to be needed in the establishment of the properties of vectors.

A wider knowledge of the nature of random variation is desirable, but the foundations of the theory are difficult. I am alarmed by the use of the expression " frequency distributions " on p. 7 of the syllabuses just issued by the Cambridge Joint Advisory Committee. There are three frequency definitions of probability, all of which have been severely criticised, and there has been no serious attempt at a reply. In fact some prominent statisticians have admitted the seriousness of the objections. " Distributions of chance " would be non-controversial. It would be a disaster if a mistaken method of approach was installed as an essential principle of elementary teaching.

Harold Jeffreys.

1838. *Examination questions.*

What are the characteristics of a good mathematical examination question, and what, if anything, have good questions, at different examination levels, in common? Consider, for instance, the University Entrance Scholarship, the Intermediate and the Final Degree Examinations. As a first provision it seems clear that some originality is desirable in both Scholarship and Degree Examination questions, out of fairness to the candidates. If the questions are museum pieces, it is possible for a candidate of good memory

to take a high place in an examination, though utterly lacking in all mathematical ability. Knowing what is written on page six of the geometry textbook is not knowing mathematics. The alternative, like that taken by the Poet to defeat the memories of the King's slaves, is to choose old questions of which the proofs are so searching and intricate that memory unsupported by understanding is almost certainly defeated.

Perhaps the most important thing of all is that the question should *teach* the candidate a piece of good mathematics. A question is a bad question, however ingenious its construction and clever its solution, if it presents a bad treatment of some aspect of the subject. This is especially relevant to the Intermediate Examination, where originality is entirely out of place (incidentally the Intermediate Examination should form an easy introduction to the final examination, and not a set of difficult exercises in school subjects).

These reflections on good and bad questions were initiated by a perusal of a recent set of examination papers for the special degree in mathematics for external students. The set contained many really excellent questions, some terrible misprints, and two questions which, for different reasons, are perhaps the worst I have ever seen. It is admittedly unfair to quote only the bad out of so many good questions, but from the commentator's point of view a bad question is good material for analysis, and a good question no material at all.

The first of these questions runs as follows (it is quoted in full):

" Prove that, if the z's are any complex numbers and c is positive, then

$$|z_1+z_2|^2 \leqslant (1+c)|z_1|^2 + \left(1+\frac{1}{c}\right)|z_2|^2.$$

Under what conditions does the sign of equality hold?

Prove that, if the a's are positive numbers such that

$$\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}=1,$$

then

$$|z_1+\ldots+z_n|^2 \leqslant a_1|z_1|^2+\ldots+a_n|z_n|^2."$$

The Lemma is clearly the particular case of the general theorem, with $n=2$, and the theorem is readily derived from the Lemma by a simple induction.

The Lemma is just the inequality

$$|z_1+z_2|^2 \leqslant \{|z_1|+|z_2|\}^2 + \{\sqrt{c}\,|z_1| - \frac{1}{\sqrt{c}}|z_2|\}^2.$$

Equality holds if, and only if, $z_2=cz_1$.

For the theorem, write $b_r=(a_{p+1}-1)a_r/a_{p+1}$, $1\leqslant r\leqslant p$, so that if $\sum_1^{p+1} 1/a_r=1$, then $\sum_1^{p} 1/b_r=1$. Assuming the theorem for $n=p$, with parameters $b_1, b_2, \ldots, b_p$, we have

$$|z_1+z_2+\ldots+z_{p+1}|^2 \leqslant (1+c)|z_1+\ldots+z_p|^2+\left(1+\frac{1}{c}\right)|z_{p+1}|^2, \text{ by the Lemma,}$$

$$\leqslant (1+c)\{b_1|z_1|^2+\ldots+b_p|z_p|^2\}+\left(1+\frac{1}{c}\right)|z_{p+1}|^2, \text{ by hypothesis,}$$

$$=a_1|z_1|^2+\ldots+a_p|z_p|^2+a_{p+1}|z_{p+1}|^2, \text{ taking } c=1/(a_{p+1}-1),$$

which completes the inductive proof.

At first sight an interesting question.

Let us, however, consider the theorem itself, ignoring the approach suggested by the Lemma.

By Cauchy's inequality,

$$(\Sigma\,|\,z\,|)^2=\left(\Sigma\frac{1}{\sqrt{a}}\sqrt{a}\,|\,z\,|\right)^2\leqslant\left(\Sigma\frac{1}{a}\right)(\Sigma a\,|\,z\,|^2),$$

which proves the theorem in a single line. What is worse, we have proved a *stronger* result than was asked for, since $|\,\Sigma z\,|^2\leqslant(\Sigma\,|\,z\,|)^2$.

In fact the inequality $(\Sigma\,|\,z\,|)^2\leqslant\Sigma a\,|\,z\,|^2$ has nothing at all to do with the complex variable, being simply a theorem for positive numbers.

It follows that *all* there is of the complex variable in the question is the inequality $|\,\Sigma z\,|^2\leqslant(\Sigma\,|\,z\,|)^2$, and the theorem stated is a *weaker* result than this expresses.

The theorem therefore amounts to the inequalities (1) $|\,\Sigma z\,|^2\leqslant(\Sigma\,|\,z\,|)^2$, (2) $(\Sigma b)^2\leqslant(\Sigma ab^2)(\Sigma 1/a)$, both of which are higher school algebra.

The second question reads thus :

" State a set of conditions sufficient to ensure that

$$\frac{d}{d\lambda}\int_0^c f(x,\lambda)\,dx=\int_0^c\frac{\partial f(x,\lambda)}{\partial\lambda}\,dx+f(c,\lambda)\frac{dc}{d\lambda}.$$

If the integral $\int_0^c\frac{f(x)\,dx}{(c-x)^{1/2}}$ is independent of c, show that $x^{1/2}f(x)$ is a constant."

In short, the student is asked to solve Abel's integral equation. Since Integral Equations are not in the syllabus of the examination (the question is not in the advanced papers), the candidate is presumably expected to show a genius for mathematics far in excess of Abel's, since that great mathematician did not obtain his solution under examination conditions.

An unkind explanation of the mystery which occurred to me is that the question was set under the mistaken impression that it admits the following simple solution.

Write $\phi(x)=x^{1/2}f(x)$, then

$$\int_0^c\frac{f(x)dx}{(c-x)^{1/2}}=\int_0^c\frac{\phi(x)}{\sqrt{cx-x^2}}\,dx=\left[\phi(x)\text{ arc sin }\frac{2x-c}{c}\right]_0^c-\int_0^c\phi'(x)\text{ arc sin }\frac{2x-c}{c}\,dx$$

$$=\frac{\pi}{2}\left[\phi(c)+\phi(0)\right]-\int_0^c\phi'(x)\text{ arc sin }\frac{2x-c}{c}\,dx.$$

Differentiating with respect to c, we find

$$\frac{\pi}{2}\phi'(c)-\int_0^c\frac{\phi'(x)}{\sqrt{cx-x^2}}\,dx-\frac{\pi}{2}\phi'(c)=0,\quad i.e.\ \int_0^c\frac{\phi'(x)}{\sqrt{cx-x^2}}\,dx=0.$$

Hence $\phi'(x)=0$, *i.e.* $x^{1/2}f(x)=\phi(x)=$ constant.

The " fallacy " lies in the last step : if $\int_0^c\psi(x)dx=0$ for all c, then we can derive $\psi(x)=0$ by a simple differentiation, *provided* $\psi(x)$ itself is independent of c.

When the integrand contains the variable c, as in the integral $\int_0^c\frac{\phi'(x)}{\sqrt{cx-x^2}}\,dx$, the deduction of $\phi'(x)=0$ is no longer elementary, but just as deep a result in the theory of Integral Equations as the solution of Abel's equation itself. (See, *e.g.*, *The Theory of Fourier Integrals*, E. C. Titchmarsh, Theorem 150.) An elementary proof is possible only on the assumption that $\phi(x)$ is a power series, the proof depending on the identity

$$\int_0^c\frac{x^r}{\sqrt{x(c-x)}}\,dx=2c^r\int_0^{\pi/2}\sin^{2r}t\,dt.$$

It is interesting to notice that if we *assume* that the equation $\int_0^c \frac{\psi(x)}{\sqrt{x(c-x)}}\,dx$ implies $\psi(x)=0$, the solution may be obtained without any differentiation.

For $\int_0^c \frac{1}{\sqrt{x(c-x)}}\,dx=\pi$, and so if a is the constant value of

$$\int_0^c \frac{f(x)\,dx}{(c-x)^{1/2}}=\int_0^c \frac{\phi(x)}{x^{1/2}(c-x)^{1/2}}\,dx, \quad \text{then}$$

$$\int_0^c \frac{\phi(x)}{x^{1/2}(c-x)^{1/2}}\,dx=\int_0^c \frac{a/\pi}{x^{1/2}(c-x)^{1/2}}\,dx,$$

i.e.
$$\int_0^c \frac{\psi(x)\,dx}{x^{1/2}(c-x)^{1/2}}=0, \text{ where } \psi(x)=\phi(x)-a/\pi, \quad \text{whence}$$

$\psi(x)=0$, *i.e.* $x^{1/2}f(x)=\phi(x)=a/\pi$, and the lemma is again totally irrelevant.

R. L. GOODSTEIN.

1839. *A note on the cycloid.*

The length of the arc of the cycloid may be obtained as the limit of a sum of circular arcs in the following very simple manner.

A regular polygon of n sides rolls on a line, the circumradius of the polygon being a. Let the vertices be numbered in order from 1 to n. Initially the side $n1$ rests on the line, then the side 12, then 23, and so on. The polygon turns about the vertices 1, 2, 3, ... in turn, turning through an angle $\frac{2\pi}{n}$ about each vertex. When the polygon is turning about the vertex r, the vertex n describes an arc of a circle of angle $\frac{2\pi}{n}$ and of radius rn. The diagonal rn subtends an angle $\frac{2r\pi}{n}$ at the centre, so that its length is $2a\sin\frac{r\pi}{n}$.

Hence the length of the path of the vertex n, in one revolution of the polygon, is $4a\sum_{r=1}^{n-1}\frac{\pi}{n}\sin\frac{r\pi}{n}$.

$$\text{But}\quad \sum_1^{n-1}\frac{\pi}{n}\sin\frac{r\pi}{n}\to\int_0^\pi \sin x\,dx=\Big[-\cos x\Big]_0^\pi=2.$$

Thus the length of an arc of the cycloid is $8a$.

The area bounded by the locus of the vertex n, and the base line, in one revolution, is the sum of the areas of the $n-1$ sectors with centres 1, 2, 3, ... , $n-1$, angle $\frac{2\pi}{n}$, and radii $2a\sin\frac{r\pi}{n}$, $r=1, 2, \dots, n-1$, together with the sum of the areas of the triangles $12n$, $23n$, $34n$, ... , $(n-2)(n-1)n$,

i.e.
$$\Sigma\frac{1}{2}\cdot\frac{2\pi}{n}\cdot 4a^2\sin^2\frac{r\pi}{n}+\frac{n}{2}a^2\sin\frac{2\pi}{n},$$

the second term being the area of the polygon.

But $\Sigma\frac{\pi}{n}\sin^2\frac{r\pi}{n}\to\int_0^\pi \sin^2 x\,dx=\frac{\pi}{2}$ and $\frac{na^2}{2}\sin\frac{2\pi}{n}=\pi a^2\frac{\sin 2\pi/n}{2\pi/n}\to\pi a^2$, and so the area bounded by an arc of the cycloid, and the base line, is

$$2\pi a^2+\pi a^2=3\pi a^2.$$

G. H. JONES.

1840. *Rotating rings of tetrahedra.*

The rotating rings of tetrahedra discovered independently by Andreas and Stalker are described by Coxeter in Ball's *Mathematical Recreations and Essays* (1939). A string of n tetrahedra is made in such a way that when it is held straight all the tetrahedra have a binary axis in common, and adjacent pairs of tetrahedra have an edge perpendicular to the axis in common. The ring formed by bringing together the two edges, one at each end of the series, is capable of rotation like a smoke-ring. The complete rotation takes place in stages: at the end of each stage the ring comes to rest of its own accord.

When n is odd, the string of tetrahedra requires to be twisted through an angle $\frac{1}{2}\pi$ in order to bring the terminal edges together to form a ring. This twist may be either right-handed or left-handed. When n is even the ring can be formed with or without twisting through an angle $t\pi$. So that no matter whether n be odd or even, the rings may exist in enantiomorphous forms.

When $n=10$ and $t=0$, the orientation of the ring is the same after two stages of rotation as it was at first; and at the end of four stages it arrives at a position of absolute coincidence. This is best seen when one face of a tetrahedron is coloured.

When $n=12$ and $t=1$ a ring can be formed, but its movement of rotation is very restricted. When $n=14$ and $t=1$ the ring can be turned completely round. In this particular case it is possible to rotate the ring in such a way that its orientation (perhaps in this case it were better to call it the *aspect*) at the end of one stage is the same as before. It will be found easier to maintain the ring in this condition when it is rotated from within outward. If one of the tetrahedra be coloured, it will be found that it progresses round the ring and arrives at its initial position (absolute coincidence) at the end of fourteen stages. Rotating an enantiomorphous pair from within outward, the coloured tetrahedron moves round the ring clockwise in one case and anti-clockwise in the other.

The completely rotatable ring with $n=16$ and $t=2$ is also interesting. It can be worked into such a position that it forms two loops with their planes at right angles to one another. The loops cross each other at an edge which is common to four tetrahedra. Each loop considered by itself has a quaternary axis of symmetry of the first order perpendicular to the plane of the loop; and the ring taken as a whole has a quaternary axis of the second order, this axis being the line of intersection of the planes of the two loops.

SIDNEY MELMORE.

1841. *The number of solutions of $x^2+y^2=z$.*

The question of the number of solutions of the equation $x^2+y^2=z^2$ in integers raised in Note 1714 can be dealt with by using complex integers.

Consider the equation $x^2+y^2=z$. It will appear later that, once we have the solutions of $x^2+y^2=z$, we can immediately obtain the solutions of $x^2+y^2=z^n$, where n is any integer.

It is known that $z\equiv 1 \bmod 4$, if z is odd, and that if z contains an even number of powers of primes $\equiv 3 \bmod 4$, they must occur as squares and be factors of x and y. We shall therefore consider them as having been removed, since they have no effect on the number of solutions.

Let $z=p_1^{\alpha_1}, p_2^{\alpha_2} \ldots p_n^{\alpha_n}$, where p_s is a prime $\equiv 1 \bmod 4$. Then it is known that there is a unique solution of the equation

$$\xi^2+\eta^2=p_s, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

apart from the order and sign of ξ and η, and there is a continued fraction process for finding ξ and η.

The solutions of $$X^2+Y^2=\mathop{\Pi}_s p_s^{\alpha_s} \qquad \text{(2)}$$

follow immediately. For

$$X^2+Y^2=(X+iY)(X-iY)=\mathop{\Pi}_s p_s^{\alpha_s}=\mathop{\Pi}_s(\xi_s+i\eta_s)^{\alpha_s}(\xi_s-i\eta_s)^{\alpha_s}.$$

Then by the uniqueness of the factorisation of complex integers, all solutions of (2) are given by

$$X+iY=\mathop{\Pi}_s(\xi_s+i\eta_s)^{\alpha_s-\beta_s}(\xi_s-i\eta_s)^{\beta_s}, \qquad \text{(3)}$$

where β_s takes all values from 0 to α_s.

We obtain solutions with $[X, Y]=1$ by taking either $\beta_s=0$ or $\beta_s=\alpha_s$. These solutions will be conjugate in pairs, so that the number of co-prime solutions is $\frac{1}{2}(2n)=2^{n-1}$.

From (3), if all the α_s are even, then $Y=0$ when $\beta_s=\frac{1}{2}\alpha_s$ for all s. Except for this case the solutions occur in conjugate pairs, which are not regarded as distinct, and consequently the total number of solutions is $\frac{1}{2}\mathop{\Pi}_s(\alpha_s+1)$.

When all the α_s are even the number of solutions is

$$\tfrac{1}{2}\{\mathop{\Pi}_s(\alpha_s+1)\pm 1\},$$

according to whether the trivial case $Y=0$ is included or not.

If instead of (2) we want the solutions of

$$X^2+Y^2=2^{\alpha_0}\mathop{\Pi}_s p_s^{\alpha_s}, \qquad \text{(4)}$$

then $$X+iY=(1+i)^{\alpha_0}\mathop{\Pi}_s(\xi_s+i\eta_s)^{\alpha_s-\beta_s}(\xi_s-i\eta_s)^{\beta_s}, \qquad \text{(5)}$$

and the number of solutions is independent of α_0 and is the same as the number of solutions of (2). The actual value of the solutions is, of course, different. For if

$$x+iy=\mathop{\Pi}_s(\xi_s+i\eta_s)^{\alpha_s-\beta_s}(\xi_s-i\eta_s)^{\beta_s},$$

then $(\pm 1\pm i)(x+iy)$ gives the pair $(x+y)$ and $(x-y)$ apart from order and sign. Thus multiplying by $(1+i)^{\alpha_0}$ does not increase the number of solutions.

In the example quoted in Note 1714,

$$x^2+y^2=1105^2=5^2\,.\,13^2\,.\,17^2,$$

we have $\alpha_1=\alpha_2=\alpha_3=2$. Hence the number of solutions is $\frac{1}{2}(3^3-1)=13$. Of these, four are co-prime and are given by

$$x+iy=(2+i)^2(3\pm 2i)^2(4\pm i)^2.$$

These are as mentioned in the note (47, 1104), (576, 943), (264, 1073) and (744, 817). There are two solutions with 5 as a common factor:

$$x+iy=5(3+2i)^2(4\pm i)^2.$$

These are (105, 1100) and (700, 855). There are two solutions with 13, and two with 17 as a common factor. These are respectively (169, 1092), (468, 1001) and (561, 952), (272, 1071). There is one solution with 5·13, one with 5·17 and one with 13·17 as common factor. These are respectively (520, 975), (425, 1020) and (663, 884). These are the thirteen solutions.

A. W. Gent.

REVIEWS.

Worked Examples in Electrotechnology. By W. T. PRATT. Pp. 262. 12s. 6d. 1945. (Hutchinson)

This work should prove of great value to the students for whom it is written and will be much appreciated by harried instructors. The setting out of the solutions in logical steps, easily to be followed, is of first-rate educational importance.

The examples are intended to cover Part I of the Associate Membership Examination of the I.E.E., and the National Certificate syllabus in Electrical Engineering. They deal fairly fully with direct current engineering, and cover alternating current circuits, single- and polyphase, including transformers, but not including the operation of alternating current rotating machinery. Except in two questions on permanent magnet moving coil instruments, measuring instruments only occur incidentally.

The D.C. machine section would be more nearly complete if examples on armature reaction, compounding, and commutation were added. It is also a pity in the reviewer's opinion that in the treatment of A.C. circuits no reference to admittance is made.

Care has been taken to maintain a high standard of accuracy, but a few small errors remain. In Example 100, page 115, it seems that the phrase "greater distance than 50 feet" should read "greater horizontal distance than 75 feet". Example 132, on page 158, has been solved for a different current from the one specified. On page 193 it is stated that "in order that a given P.D. may be developed across any part of the circuit with minimum supply P.D., the circuit must be in resonance, *i.e.* the condenser reactance must be equal to the coil reactance". This is not so, except for the P.D. across the total resistance of the circuit. In the question the P.D. across the condenser is required, and the misstatement leads to an error of the order of 7 per cent. Figure 147, on page 248, may be misleading, as it is inconsistent with the following figures also representing the equivalent circuit of a transformer. In these, separate paths for the components of the magnetising current are shown together with the two coils of a "perfect transformer", whereas in Fig. 147 the wattless component of the no-load current is made to flow in the primary of the "perfect transformer". One or two small printer's errors have been overlooked, as, for instance, the omission of a letter in line 11 on page 117, the omission of π in line 16 of page 169, and the displacement of an index in the definition of the prefix, micro-micro-, on page 19.

These points are mentioned in anticipation of a reprint, and are small blemishes in a book which can be warmly recommended. G. F. N.

Theory of Games and Economic Behaviour. By JOHN VON NEUMANN and OSKAR MORGENSTERN. Pp. xviii, 625. 66s. 6d. 1944. (Princeton U.P.; Humphrey Milford)

This book is full of new and important ideas.

The mathematical theory of games in the past consisted chiefly of the application of mathematics to special games or kinds of games. This book, on the other hand, analyses the idea of a game in its most general form, and also the idea of a solution, and actually determines the solution in many important simple cases.

Probably no general definition of a "game" has been given before, taking into account the possibilities of mixtures of chance moves and moves by players, varying amounts of information about the previous moves, and so on. So that is the first task the authors achieve. A specially important

classification is into " zero-sum " games, in which money (or other measure of " utility ") simply changes hands at the end of the game, and " non-zero-sum " games, in which money may be created or destroyed.

The next task is to define the solution of the game. Here even in the simplest case of a zero-sum two-person game we run into difficulties. For example, consider the game in which two players Alf and Bill each place a coin on a table unseen by the other. If on comparing the coins they agree (both heads or both tails), Alf claims 1d. from Bill ; if they disagree, Bill claims 1d. from Alf. What are their best ways of playing? Clearly if Alf tosses his coin on to the table, he will ensure himself an even chance of winning whatever Bill does. And that is the best he can do, for Bill in the same way can ensure himself an even chance of winning. Neither, therefore, can do better than choose his position by chance. This sort of solution, in which strategy is partly decided by chance, was discovered independently by R. A. Fisher, but in this book it is shown to hold for all two-person games with zero sum. " Bluffing " in poker—occasional high bids on a low hand—is of this nature.

Quite different difficulties come in with games of more than two players. Generally it is of advantage for sets of players to combine together against the remainder. (When no combinations offer any advantage the game is called " inessential " and is relatively trivial.) To maintain such a combination one player or set of players may buy the support of other players, so that the final distribution of money may be changed considerably from that prescribed by the rules of the game. Such a (possibly changed) distribution is called an " imputation ". A " solution " one would expect to be an imputation which is in some way better than all others—but in what senses " better "?

An obvious answer is that if on considering two imputations α and β, we find a set of players in whose interest it is to have imputation β rather than α, and who can be sure of getting it (or its equal) no matter what the other players do, then we might say that β is " better " than α, or in the authors' terminology, β " dominates " α. If we could find an undominated imputation α, it would not be to the interest of any group to change from it, and so it would form a natural solution. But unfortunately it is only possible in an inessential game.

Because of these difficulties, the authors suggest the following definition of *a* solution (in general, a game has many such solutions of differing types).

" A solution is a set V of imputations which between them dominate all imputations not in V, but no others ." A brief justification of this is hardly possible, but it is shown that it represents a sort of stability based on compromise. This is specially important for economics whose negotiations, combines, buying, etc., so much resemble what goes on in a game with many players that it is suggested that this theory will lead to the first satisfactory mathematical theory of economics. If an economic system owes its stability to compromise, what of our political and ethical standards of behaviour?

The complete set of solutions is known in general only for the two- and three-person games : but certain special types of more complicated games are exhaustively studied, with interesting and unexpected results. For example, if two entirely separate games played by distinct sets of players are considered formally as one game, there are solutions in which money is passed from one game to the other. And a combination of players need not claim all the money they might from their opponents. The theory does not yet include games in which it is forbidden to buy the support of other players, but that is one suggested generalisation.

The mathematics throughout the book is entirely modern in spirit, using matrices, theory of sets, axiomatic methods, etc., but the treatment is every-

where elegant and elementary, and does not assume a prior knowledge of these methods. The explanations are careful and detailed, and when attacking a difficult problem the authors have an admirable method of first giving a lucid account of the strategy they will adopt. The printing is good, but the following errors might be confusing :

p. 95, ll. 6, 8. *For* " field " *read* " element ".
p. 109, l. 19. *For* second $K_1(\tau_1, \tau_2)$ *read* $K_1(\tau_2, \tau_1)$.
p. 161, l. 18. *For* " strictly determined " *read* " specially strictly determined ".
p. 412, l. 5. *For* " since s, q " *read* " since s, p ".
p. 514, l. 8. Insert − before first Σ.
p. 525, l. −2. *For* " old " *read* " new ".

C. A. B. SMITH.

Tables of Elementary Functions. By F. EMDE. Pp. xii, 181. $3.20. 1945; lithoprinted from the edition of 1940. (Edwards, Ann Arbor, Mich. ; Scientific Computing Service, 23 Bedford Square)

For the third (1938) edition of Jahnke und Emde, *Funktionentafeln*, the 75 pages of elementary matter were sacrificed, and Dr. Emde promised a separate publication. This appeared in 1940 but has not, of course, been available here ; the present version is a lithoprint by Messrs. Edwards Bros., under the authority of the U.S. Custodian of Enemy Property. The work has been well done, and the volume is neat, easily legible, and well turned out, if a trifle expensive.

There is much more in this volume than there was in the early pages of Jahnke-Emde, though the main ground covered is the same. Roughly, it deals with powers and with the exponential function ; the latter of course includes the logarithm, the circular and hyperbolic functions, and certain special combinations, of value in applied mathematics and engineering.* There are also sections on the manipulation of complex numbers, for example, tables for the modulus and argument of the sum of two given complex numbers ; and sections on the numerical solution of algebraic equations of degrees 2, 3, and 4. As in the parent work, the text is in parallel German and English, there are many diagrams, including relief diagrams, and much helpful material in the way of formulae.

The final section gives hints on computation, and a useful but uncritical list of books on numerical mathematics. It is in this section that Dr. Emde has been least happily served by his translator. " Only for the functions $e^x, \ldots \cos x$ results the constant interval $h = 0{\cdot}02$ rad. " is not English, though it can be forgiven. But there are places where the meaning becomes obscured in translation : " For a numerical calculation in a series it is not the most important whether it be convergent or divergent. A series is useful for numerical calculation, if its beginning is convergent." Why not stop at the first term?

This apart, the new Emde is a valuable addition to our shelf of tables ; it is worth a place in the school library. T. A. A. B.

Sechsstellige Trigonometrische Tafel. By H. BRANDENBURG. Pp. xxiv, 304. $5. 1945 ; lithoprinted from the edition of 1932. (Edwards, Ann Arbor, Mich. ; Scientific Computing Service, 23 Bedford Square)

This lithoprinted edition of the well-known six-figure tables prepared by Brandenburg appears under the same auspices as the Emde volume reviewed above. The volume is clearly reproduced and neatly bound.

* It may be remarked that extensive use is made of the right-angle as an angular unit.

The main table gives the sines, cosines, tangents and cotangents from 0° to 45° at intervals of 10″. This of course covers the quadrant conveniently. First differences are provided, and the proportional parts of the differences on a page are given in a margin to the table on that page. Auxiliary tables provide the cotangent for the first three degrees for every second, and the sines and tangents for the first degree at intervals of 10″, as in the main table, but in this case six significant figures are given, irrespective of the number of zeros following the decimal point.

Dr. Comrie has supplied a list of eight errors, not all serious, and this has been inserted ; so short a list from so severe a critic indicates a high standard of reliability.

The tables are convenient for use. Aesthetically they fall short of elegance by what seems to me to be too frequent a use of " rules ", giving the page a crowded, jig-saw appearance ; but this will not be considered a fault by all table-users, and matters little since the fount is of clear, old-style figures.

T. A. A. B.

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette*.

Dear Sir,—A personal reference by Professor G. H. Livens, on p. 10 of the February *Gazette* affords me an opportunity of paying a tribute to the late Mr. William Welsh of Jesus College ; an opportunity which I cannot resist. Mr. Welsh was a very great teacher, and I have always considered myself fortunate to have been one of his pupils. He succeeded Mr. William Walton (eighth wrangler, 1836) as lecturer at Magdalene, in my first term of residence. As we were not then combined with any other College for lectures, he had the difficult task of covering the schedule for the tripos (the old Part I) by giving the men of each year three lectures weekly, each lecture lasting only 45 minutes. His method, which worked well with a small class, was to sit at a table surrounded by his class and write out his lecture (using no notes) in an easily legible hand, at the same time dictating what he wrote, and at the end giving what he had written to one of the class. Though he covered a vast amount of ground, there was not time for everything ; and I have no doubt that Professor Livens got more from him at Jesus College, some twenty years later, including the interesting application of the method of energy to problems of changing mass, which I had not seen until I read the February *Gazette*.

Yours faithfully, A. S. Ramsey.

THE INTEGRAL DEFINITION OF THE LOGARITHM.

To the Editor of the *Mathematical Gazette*.

Dear Mr. Editor,—Will you allow us to comment on Mr. Tuckey's note in the *Gazette* of February, 1945?

No doubt it is true that Hardy's *Pure Mathematics* (1st edition, 1908) has been a strong influence in favour of the integral definition of the natural logarithm. But Prof. Hardy would be the last to claim either priority or a monopoly in this matter. T. J. I'A. Bromwich, in the preface to his *Infinite Series* (1907), writes : " It will be noticed that from Art. 11 onwards free use is made of the equation $\dfrac{d}{dx}\log x = \dfrac{1}{x}$, although the limit of $\left(1+\dfrac{1}{n}\right)^n$ from

which this equation is commonly deduced is not obtained until Art. 57. To avoid the appearance of reasoning in a circle, I have given in Appendix II a treatment of the theory of the logarithm of a real number starting from the equation $\log a = \int_1^a dx/x$. The use of this definition of a logarithm goes back to Napier, but in modern teaching its advantages have been overlooked until comparatively recently."

Bromwich adds references to a paper by Bradshaw dated 1903 and to Osgood's *Lehrbuch der Funktionentheorie*. Reference may be made here also to the English translation of Felix Klein's *Elementary Mathematics from the Advanced Standpoint* (analysis volume, p. 155).

We mention this as a matter of historic interest. It does not of course affect Mr. Tuckey's argument.

The teacher who accepts the recommendation to use the integral definition, without conviction, merely from deference to the authorities who have made it, and does not otherwise alter his teaching arrangements, will probably find himself in the ridiculous position suggested by Mr. Tuckey of saying to his pupils: Come, now, let's pretend that we cannot differentiate $\log x$. This will be because he has proceeded too far in his course of differential calculus before starting integration. The serious objection to Mr. Tuckey's differentiation of a^x is not at all that it requires more intuitive ideas about the gradient of a curve, but that a^x is being differentiated simply because the teacher is scratching about for something to differentiate: not because he wants to differentiate it in connexion with some problem that arises naturally in a logical or a practical course of mathematics.

When Mr. Tuckey says that differentiation comes before integration, he cannot mean that *all* differentiation comes before *any* integration, although that may have been approximately true in his school-days and ours. But nowadays the differentiation that precedes applications of integrals to areas and volumes is generally limited to powers of x. It need not even include the differentiation of $\sin x$, although the trigonometrical functions are more likely to arise in elementary work than logarithmic or exponential functions. Thus the pupil is brought, at an early stage, up against the missing link in the formula

$$\int x^n\,dx = x^{n+1}/(n+1), \quad n \neq -1,$$

and it is then easy and can be exciting to investigate $\int_1^a dx/x$.

Another disadvantage of proceeding too far with differentiation is that it tempts the pupil, when he eventually reaches integrals, to regard integration as a purely tentative process to be carried out by guessing when such and such an answer was obtained to a differentiation. Although it is true that there is a tentative side to integration, it is far more important for the pupil to realise that there is a systematic side. Prof. Hardy's influence has also been in this direction, notably in his tract on *Integration* in the Cambridge series of tracts.

It must be admitted that the integral approach to logarithms loses its interest if the pupil knows the answer beforehand. But he ought not to know it. The ideas on which the method is based are not difficult, and should be introduced when the integral of $1/x$ or the natural logarithm first appears.

Nobody can describe the method which Mr. Tuckey advocates as exciting, and at this stage it cannot be given any purely theoretical support. On the other hand, the integral method, apart from its interest, also serves as an

introduction to an important form of mathematical procedure, and so it appears to us to possess just that "outlook value" that should be a dominating consideration in the choice of subject-matter for mathematical sixth forms. Yours, etc., C. V. DURELL and A. ROBSON.

POETS' CORNER.

To the Editor of the *Mathematical Gazette*.

SIR,—For some time I have collected specimens of verse (mostly light) on mathematical and astronomical themes. I should be glad if you could spare a little space to enable me to ask members of the Association to help in retrieving scattered masterpieces. It will be enough to give references to the books or journals in which they can be found, and the smallest contribution will be gratefully acknowledged. Yours faithfully, A. P. ROLLETT,

4 Oak Lane, Sevenoaks, Kent.

TEXTBOOKS AS TEACHERS.

To the Editor of the *Mathematical Gazette*.

SIR,—Would that they were! I speak of university student textbooks. Roughly, these are either class-books, designed as aids to the training of examinands in classes; or they are lectures without a lecturer.

In no case that I know of does any one of them, modern or not, solicitously regard and properly provide for the needs of the learner when he is alone with his books. Yet during that part of his learning time the learner has no help within reach except his books. And it is then that there occurs, for very many learners, an appalling waste of their time.

Responsibility for this wrong done to them divides between authors and publishers. A textbook, like a teacher, is under obligation to answer every question which a learner legitimately puts to it. Further, he must not be needlessly delayed in getting his answer.

I have not space to set out separately here the multiple unnecessary compulsions to waste of time enforced upon a learner in solitude by every one of these textbooks. Each separate unnecessary one of these is in itself a wrong. The accumulated total of them piles up to a great evil. This evil is the more cruel in that the injuries which it inflicts are not merely according to weakness, but become harsher in steeply rising proportion to it. The nature and the hurt of each of these injuries I know by experience. The feelings of the weak concerning them I know from within, and I will voice them.

I am ready to justify my indictment by as many instances as may be desired from well-known current textbooks. The worst of the injuries arise from gross neglect of the mere mechanism of presentation, especially from almost unbelievable deficiency in: (1) references back to the book itself; (2) indexes; (3) reference to ancillary books. The multitude of other injuries I must leave unclassified and uncounted.

Finally, and it would be my best contribution, I believe that I can point out a remedy that would be efficacious, unobjectionable to authors, and yet *cheap* to publishers. Yours, etc., OWEN MADDEN.

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